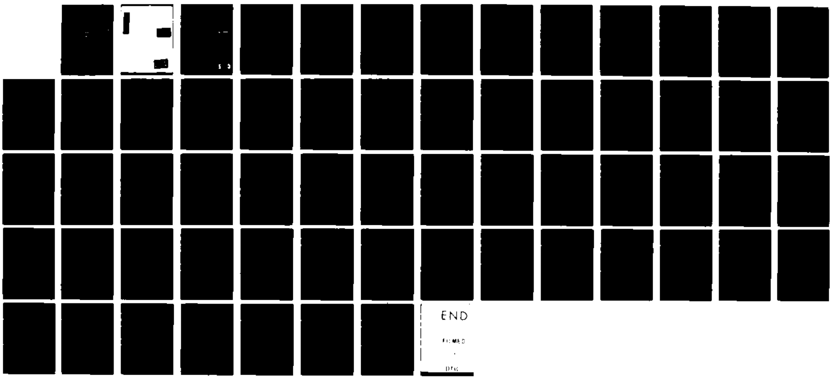
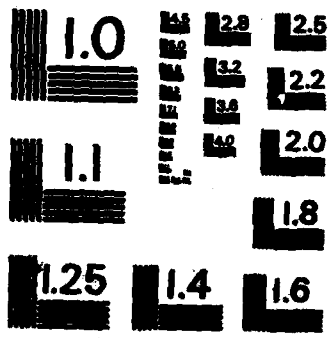
AD-A121 804 INFORMATIONAL EQUILIBRIUM(U) STANFORD UNIV CA INST FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES R KAST SEP 82 TR-382 N00014-79-C-0685 1/1

UNCLASSIFIED F/G 5/2 NL





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD A121804

INFORMATIONAL EQUILIBRIUM

by

Robert Kast

Technical Report No. 382
September 1982

Prepared Under
Contract ONR-N00014-79-C-0685, United States Office of Naval Research
Center for Research on Organizational Efficiency
Stanford University

THE ECONOMICS SERIES
INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
Fourth Floor, Encina Hall
Stanford University
Stanford, California
94305

INFORMATIONAL EQUILIBRIUM*

by

Robert Kast**

1. Introduction

Decision making over time in an uncertain environment has motivated much research, and given rise to different definitions of equilibria. Research on Temporary Equilibria concentrated on the consistency of agents when making forecasts, together with classical rational behavior under risk. Rational Expectation Equilibria were developed from the seminal paper of Muth until the model of Anderson-Sonnenschein provided an existence theorem: an equilibrium exists if agents make decisions which generate through equilibrium market price, the probability distribution they used in making their decisions. We call Informational Equilibrium an equilibrium of decisions and forecasts of agents in a dynamic process: agents use their forecasts to make their decisions and these decisions generate through the system a future distribution which matches the forecasts. The structure of the model is quite general and is not specially related to a sequence of markets.

---

*This work has been sponsored in part by Office of Naval Research Contract ONR-N00014-79-C-0685 at the Center for Research on Organizational Efficiency at Stanford University, and in part by a grant from Minetère Francais des Affaires Etrangères, for the year 1980-81. I want to acknowledge here for their help and comments, Professors K.J. Arrow, P.J. Hammond and M. Kurz from Stanford University, and Robert Anderson and J.P. Florens, who helped me through mathematical difficulties during their visit, as well as the clerical staff in the Department of Economics at Stanford.

**Department of Economics, the University of Southampton, Highfield, Southampton S09 5NH, England.

The sequence of decisions of agents and the sequence of states of the market form a stochastic process that we refer to as the "model" in which agents decide. It follows Blackwell's [1965] model in that it assumes a Motion Law (relating decisions at time t and the distribution at time $t+1$) as given. We assume that our agents behave consistently with the model, in the sense that they know its structure if not its exact distributions. Then they use the information they receive about the state of the system to forecast the future states, their forecasts being "parallel" to the unknown Motion Law.

At each period (time is discrete) they decide on a present action and on a plan for the future. We do not focus here on the consistency that a temporary equilibrium would require (in the sense of Grandmont for example).

The model is presented in Section 2. The definition of equilibrium was given by Shefrin [1980], namely it is a fixed point of a couple of correspondences: the correspondence relating decisions of agents and the distribution they generate next period, and the correspondence relating forecasts of agents and their resultant decisions.

In Section 3 we show that the model has "good" properties, because the distribution generated next period is a "continuous" function of the decisions of agents. This is a necessary condition, which is verified by this model.

In Section 4 we give two examples in which the decisions of agents are continuous function (loosely speaking) of their forecasts

(i.e. in which agents decide in such a way that a temporary equilibrium occurs as specified by the Grandmont and Nash models).

As we make "heavy" use of probability transitions, we had to develop algebraic properties of this mathematical tool (in Appendix 1). The fixed-point property defining our equilibrium relies on topological properties of transitions which have been established in Appendix 2. Appendix 3 is a very limited attempt to explain what a forecast could be. Clearly a complete theory of rational behavior of agents (concerning the use of forecasts) should be developed.

2. The Model

In short, if H^t is the set of states of the system, if agent i knows only a signal in S_i^t by a random variable σ_i^t, if he has a forecast Ψ_i^{t+1} on the future state of the system which he uses to make his decision α_i^t, then we prove:

(1) That there exists a Markov kernel Q^t relating H^t and H^{t+1}, which is a function of the decision α_i^t, $i \in I$.

(2) That there exists a "posterior" distribution M_i^t on H^t which is a function of α_i^t: M_i^t.

Thus the distribution on H^{t+1} generated by the α_i's, $Q^t \circ M_i^t$, can be compared to the distribution that agent i forecasts, Ψ^{t+1}. We shall use a "fixed-point" argument, where an informational equilibrium would take place when the two distributions match. The

conditions for the fixed-point theorem to hold depend on the way actions are defined from the forecasts. We shall see that for various standard types of equilibria, they hold. In particular, if one uses the temporary equilibrium framework one can use the standard existence results to obtain existence of an informational equilibrium.

2.1 Process of Decision Making Under Uncertainty

Framework:

(1) I is a set of decision makers. Let us suppose I finite unless otherwise stated. Time is discreet: $t \in \mathbb{N}$.

(2) States of nature are generated by a process $(\Omega^t, \mathcal{O}^t, \nu^t)$, $t \in \mathbb{N}$, unknown to the agents. We suppose (for mathematical purposes) Ω^t to be metric and separable and σ^t to be its Borel sigma-algebra.

(3) At each time t, agent i chooses an action for time t from a set A_i^t (which is assumed metric separable) and its sigma-algebra is the set of Borel-subsets. Let us call $A^t = \underset{i \in I}{\times} A_i^t$. A^t has the same properties as A_i^t. Let us use the following notation, $E^{t+} = \underset{s \geq t}{\times} E^s$ and $E^{t-} = \underset{s \leq t}{\times} E^s$.

(4) States of the systems at time t are elements of $\Omega^{t-} \times A^{t-1^-} = H^t$. Thus $h^t \in H^t$ is the sequence of the previous and present exogenous states of nature and the previous actions: $h^t = (w^0, w^1, \ldots, w^t, a^0, \ldots, a^{t-1})$. (That is, h^t is the history until t.) Let us call μ^t the distribution induced on H^t from Ω^t and the

decision process. We suppose that states at time t are related to states of nature at time $t + 1$ by the Motion Law.

$$\pi^{t+1}: H^t \times A^t \longrightarrow \Omega^{t+1} .$$

This means that for every history up to t and actions of agents at time t there is a distribution on the sets of future outcomes.

(5) Each agent i does not know Ω^{t^-}, nor maybe all of A^{t-1^-}, but we suppose that he receives a signal which is a random variable, $\sigma_i^t: H^t \longrightarrow S_i^t$, where S_i^t is a metric separable space and is measurable with its Borel sigma-algebra. We also assume that he has a "forecast function." More generally we suppose that, from this signal and his action, he can forecast a probability on the space of his future signals by: $\psi_i^{t+1}: S_i^t \times A_i^t \longrightarrow S_i^{t+1^+}$. (For a justification of ψ_i^{t+1}, see Appendix 3.) Remember that there is no restrictive assumption on $S_i^{t+1^+}$, so that although the agent is going to receive only a signal S_i^{t+1}, he could well have to forecast on a much broader space like H^{t+1} itself.

(6) Each agent i makes a decision at each time t. It does not only consist of an action $a_i^t \in A_i^t$; agent i chooses a <u>decision rule</u> which is a transition from his future signal to his future actions. A decision rule is defined by: $\forall\, s_i^t \in S_i^t,\ \alpha_i^t(s_i^t) = (a_i^t(s_i^t), \alpha_i^{t+1}(s_i^t))$ where $a_i^t(s_i^t) \in A_i^t$ and $\alpha_i^{t+1}(s_i^t)$ is defined by $\forall\, s_i^t \in S_i^{t+1}$,

$\alpha_i^{t+1}(s_i^t)(s_i^{t+1}) = (a_i^{t+1}(s_i^{t+1}), \alpha_i^{t+2}(s_i^{t+1}))$, etc., so that for any $(s_i^t, s_i^{t+1}, \ldots) \in S_i^{t^+}$ the result of the decision rule α_i^t would be

$$(a_i^t(s_i^t),\ a_i^{t+1}(s_i^{t+1}),\ldots) \in A_i^{t^+} .$$

Therefore the agent decides what may be an infinitely long sequence of actions each dependent upon the stochastic evolution of the process and his previous actions. Such an extensive calculation is of course difficult to handle and indeed it is difficult to envisage an agent undertaking it. We shall therefore simplify by assuming that agents have in mind a stationary decision rule. This does not mean that their rule will indeed be unmodified over time, but they reduce their calculation to manageable proportions by assuming this will be the case. Thus their assumed rule will be

$$\alpha_i^t = (a_i^t,\ \alpha_i^{t+1})$$

with

$$\alpha_i^{t+1} = \alpha_i^t .$$

Hence we can consider α_i^t as

$$\alpha_i^t:\ S_i^t \longrightarrow A_i^t .$$

To summarize the decisions of agents we shall call $\alpha^t:\ H^t \longrightarrow A^{t^+}$, a transition such that the induced transition from α^t by projection on A_i^t is $\alpha_i^t \circ \sigma_i^t$

$$\begin{tikzcd} & S_i^t \arrow[rr, "\alpha_i^t"] & & A_i^t \\ H^t \arrow[ur, "\sigma_i^t"] \arrow[rrr, "\alpha^t"'] & & & A^t \arrow[u, dashed, "\mathrm{proj}"'] \end{tikzcd}$$

$A_i^t \quad A^t$

α^t could be $\prod_{i\in I} \alpha_i^t \circ \sigma_i^t$, the meaning then being an aggregation of decisions of agents when these decisions are stochastically independent (relative to the distribution induced on A^{t^+}). Other aggregating processes could be used according to specific models.

(7) Each agent i makes his decision by maximizing a utility function. To set it in the most general framework this function should be dependent on the present signal and on the decisions of other agents:

$$U_i^t(s_i^t,(a_j^t)_{j\neq i}, \alpha_i^t(s_i^{t^+})) \ .$$

But as $\alpha_i^t(s_i^{t^+})$ is a function of the unknown future signals, agent i uses his forecast function to compute his expected utility:

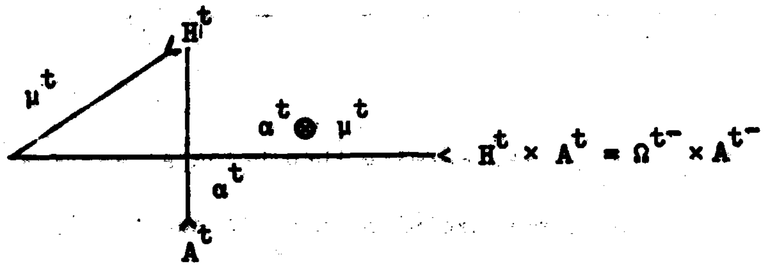
$$V_i^t(s_i^t,v_i^t) = \int_{S_i^{t+1^+}} U_i^t(s_i^t,(a_j^t)_{j\neq i}, \alpha_i^t(s_i^t,s_i^{t+1^+}) \ ,$$

$$V_i^{t+1}(s_i^t,a_i^t(s_i^t))ds_i^{t+1^+}) \ .$$

2.2 Process of the States of the System

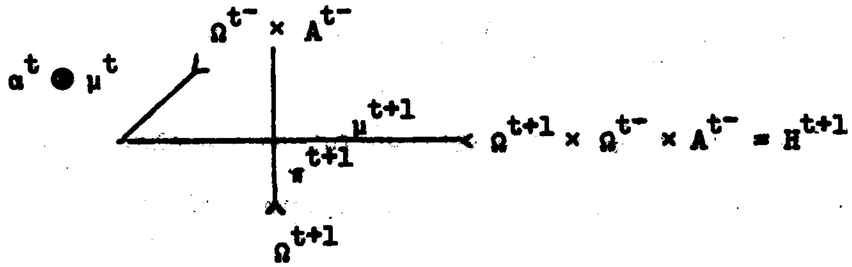
The process (H^t,μ^t) represents the sequence of states of the

system $H^t = \Omega^{t-} \times A^{t-1^-}$. When the aggregate decision α^t: $H^t \multimap A^t$ is taken it induces a distribution on A^t (that we still write α^t) and thus on $\Omega^{t-} \times A^{t-} = H^t \times A^t$; naturally it is $\alpha^t \otimes \mu^t$. (See Appendix 1 for notation and definition of the cross product.)



Now the Law of Motion of the system is π^{t+1}: $H^t \times A^t \multimap \Omega^{t+1}$ and we define a distribution on $\Omega^{t+1} \times H^t \times A^t = H^{t+1}$ by

(2.2.1) $\qquad \pi^{t+1} \otimes (\alpha^t \otimes \mu^t) = \mu^{t+1}$



The process is defined by the recurrence formula (2.2.1). To be a Markov process, a transition Q^t: $H^t \multimap H^{t+1}$ must exist, such that:

(2.2.2) $\qquad \mu^{t+1} = Q^t \circ \mu^t$.

(See Appendix 1 for an equivalent definition of a Markov process.)

Q^t does not appear on the diagrams but we can extract it from the defining formula. μ^{t+1} is defined on $H^{t+1} = \Omega^{t+1} \times H^t \times A^t$ by $\forall\ \bar{\bar{H}}^{t+1} = \bar{\bar{E}}^{t+1} \times \bar{A}^{t-} \in \mathcal{O}^{t-} \otimes A^{t-}$ where $\bar{\bar{E}}^{t+1-} \times \bar{A}^{t-} = \bar{\bar{E}}^{t+1} \times \bar{\bar{H}}^t \times \bar{A}^{t-}$,

$$\mu^{t+1}(\bar{\bar{E}}^{t+1-} \times \bar{A}^{t-}) = \int_{\bar{\bar{E}}^{t-} \times \bar{A}^{t-}} \pi^{t+1}(w^{t-}, a^{t-}, \bar{\bar{E}}^{t+1})(\alpha^t \otimes \mu^t)(dw^{t-}, da^{t-}). \tag{2.2.3}$$

As $\alpha^t \otimes \mu^t$ is defined by: $\alpha^t \otimes \mu^t(\bar{\bar{H}}^t \times \bar{A}^t) = \int_{\bar{\bar{H}}^t} \alpha^t(h, \bar{A}^t)\mu^t(dh)$,

we have

$$\mu^{t+1}(\bar{\bar{E}}^{t+1-} \times \bar{A}^{t-}) = \int_{\bar{\bar{E}}^{t-} \times \bar{A}^{t-}} \pi^{t+1}(w^{t-}, a^{t-}, \bar{\bar{E}}^{t+1})\, \alpha^t(h, da^t)\, \mu^t(dw^{t-}, da^{t-}) \quad . \tag{2.2.4}$$

If $\mu^{t+1} = Q^t \circ \mu^t$ we would have

$$\mu^{t+1}(\bar{\bar{E}}^{t+1-} \times \bar{A}^{t-}) = \int_{H^t} Q^t(h^t, \bar{\bar{E}}^{t+1-} \times \bar{A}^{t-})\mu^t(dh) \quad . \tag{2.2.5}$$

We can write the formula (2.2.4) such that, compared with (2.2.5), we see that Q^t is the expression between brackets:

$$\mu^{t+1}(\bar{\bar{E}}^{t+1-} \times \bar{A}^{t-}) = \int_{H^t} \left[\chi_{\bar{\bar{E}}^{t-} \times \bar{A}^{t-}}(h) \int_{\bar{A}^t} \pi^{t+1}(w^{t-}, a^{t-}, \bar{\bar{E}}^{t+1})\alpha^t(h, da^t) \right] \mu^t(dh) \quad .$$

Thus, Q^t is defined on the square sets (and its extension on H^{t+1} is unique) for μ^t almost every $h^t \in H^t$ and for every $\bar{\bar{H}}^{t+1} = \bar{\bar{E}}^{t+1} \times \bar{\bar{H}}^t \times \bar{A}^t \in H^{t+1}$:

$$Q^t(h^t, \bar{\bar{H}}^{t+1}) = \chi_{\bar{\bar{H}}^t}(h^t) \int_{\bar{A}^t} \pi^{t+1}(h^t, a^t, \bar{\bar{E}}^{t+1})\alpha^t(h^t, da^t) \quad . \tag{2.2.6}$$

Thus, we have established the following.

Proposition: The process defined by a measure μ° on H°, the sequence $H^{t+1} = \Omega^{t+1-} \times A^{t-} = \Omega^{t+1} \times H^{t} \times A^{t}$, the Motion Law π^{t+1}: $\Omega^{t-} \times A^{t-} \longrightarrow \Omega^{t+1}$, and the aggregate decision rule α^{t}: $H^{t} \longrightarrow A^{t}$, is a Markov process whose kernel Q^{t} is defined above by (2.2.6). The underlying process $\Omega^{t} \times A^{t-1-}$ is more general as its distribution depends on all previous states, but of course if it were assumed Markovian from the beginning, as is often done, H^{t} also would be Markovian, a forticri.

Very often in similar models, the exogenous stochastic process (Ω^{t},ν^{t}) is assumed to be a Markov process. The fact that the state of the system is a Markov process is then a direct consequence as we have shown elsewhere for the models of Grandmont and Hildenbrand [1974], and Green and Majumdar [1972] (see Appendix 4).

2.3 The Decision Process of Each Agent

At time t, agents know only the previous signals of S_i^{t-} and previous actions of A_i^{t-}. In another context we could consider that he would know past signals and actions of other agents, but here let him only deal with the set S_i^{t} of signals he receives from previous states and actions by way of the random variable:

$$\alpha_i^t: \; H^t \longrightarrow S_i^t \; .$$

For every signal and for each of his actions, agent i has a forecast of what his future signals will be $\Psi_i^{t+1}: S_i^t \times A_i^t \longrightarrow S_i^{t+1^+}$, from the structure

$$\begin{tikzcd}
 & & H^t \arrow[dr, "\sigma_i^t"] \arrow[ddd, "\alpha_i^t \otimes \sigma_i^t"] & & \\
 & A_i^t & & S_i^t \arrow[ll, "\alpha_i^t"'] & \\
 \cdot \arrow[uurr, "\mu^t"] \arrow[rrrr, dashed, "\alpha_i^t \otimes \sigma_i^t \otimes \mu^t"] & & & & H^t \times S_i^t \times A_i^t \\
 & & S_i^t \times A_i^t & &
\end{tikzcd}$$

We deduce on $H^t \times S_i^t \times A_i^t$ the distribution $(\alpha_i^t \otimes \sigma_i^t) \otimes \mu^t$. Now the problem of forecasting for agent i is that of comparing his forecast $\Psi_i^{t+1}(s_i^t, a_i^t)$ with the distribution on S_i^{t+1}. This distribution is induced from the distribution on H^{t+1} by the signal σ_i^{t+1}.

$$\begin{tikzcd}
H^t \arrow[d, "Q^t"] & & S_i^t \times A_i^{t^+} \arrow[d, "\Psi_i^{t+1}"] \\
H^{t+1} \arrow[rr, "\sigma_i^{t+1}"] & & S_i^t
\end{tikzcd}$$

We clearly need a link from $S_i^t \times A_i^{t^+}$ to H^t so that its composition with $\sigma_i^{t+1} \circ Q^t$ can be compared with Ψ_i^{t+1}.

<u>Proposition</u>: For every decision process α_i^t, there exists a transition M_i^t from $S_i^t \times A_i^{t^+}$ to H^t which is defined by

$$(2.3.1) \qquad M_i^t \otimes [(\alpha_i^t \otimes \sigma_i^t) \circ \mu^t] = (\alpha_i^t \otimes \sigma_i^t) \otimes \mu^t \quad .$$

Existence comes from the theorem of Jirina when the spaces H^t and $S_i^t \times A_i^{t+}$ are both metric and separable; this result is fundamental in Bayesian inference: M_i^t is the posterior distribution.

We can now clearly define what we mean by informational equilibrium: the expectation Ψ_i^{t+1} of agent i at time t, must match the objective transition

$$(2.3.2) \qquad \sigma_i^{t+1} \circ Q^t \circ M_i^t \quad .$$

2.4 Informational Equilibrium

1. From formula (2.2.6) we see explicitly how Q^t is a function of α^t. Let us write $Q^t(\alpha^t)$ for clarity. From formula (2.3.1), we see that M_i^t is a function of α_i^t. The relation between the family of α_i^t's and the distribution they generate on S_i^{t+1} next period is then

$$\beta_i(\alpha^t) = \sigma_i^{t+1} \circ Q(\alpha^t) \circ M_i^t(\alpha_i^t)$$

where $\alpha^t = (\alpha_i^t)_{i \in I}$ is an element of the set $\mathcal{D} = \overset{n}{\underset{i=1}{\mathsf{X}}} \mathcal{D}_i$ the product of the sets of decisions of each agents $\mathcal{D}_i = \{\alpha_i^t \circ \sigma_i^t : \ s_i^t \longrightarrow A_i^t\}$.

The relation between Ψ_i^{t+1} and α_i^t depends on the specific model of decision making. In models where actions are defined by a temporary equilibrium for example, for every Ψ_i^{t+1} verifying certain

conditions, the corresponding α_i^t's will be in the set of optimal decisions. Let us call $\gamma_i(\psi_i^{t+1})$ the set of decisions that agent i makes according to his forecast.

If F_i is the set of forecasts of agent i, $F_i = \{\psi_i^{t+1}: S_i^t \times A_i^t \longrightarrow S_i^{t+1}\}$, γ_i is a correspondence from F_i to $\mathcal{D}_i$.

Let us call $\mathcal{D} = \underset{i=1}{\overset{n}{\times}} \mathcal{D}_i$ and $F = \underset{i=1}{\overset{n}{\times}} F_i$ and

$$\beta: \mathcal{D} \longrightarrow F$$

$$\alpha^t = (\alpha_i^t)_{i\in I} \longmapsto \beta(\alpha^t) = (\beta_i(\alpha^t))_{i\in I}$$

and

$$\gamma: F \longrightarrow \mathcal{D}$$

$$\psi^t = (\psi_i^t)_{i\in I} \longmapsto \gamma(\psi^t) = (\gamma_i(\psi_i^t))_{i\in I} \quad .$$

We then have

Definition 1: The families of forecasts ψ^t and of decisions α^t are in informational equilibrium if:

$$\beta(\alpha^t) = \psi^t \quad \text{and} \quad \gamma(\psi^t) = \alpha^t \quad .$$

Or, if we consider the correspondence

$$(\beta,\gamma): \mathcal{D} \times F \longrightarrow F \times \mathcal{D} \quad ,$$

we have

<u>Definition 2</u>: An informational equilibrium is a family of decisions $\alpha^t = (\alpha_i^t)_{i\in I}$ and of forecasts $\psi^t = (\psi_i^t)_{i\in I}$ such that (α^t,ψ^t) is a fixed-point of the correspondence (β,γ).

Existence of such a fixed-point relies on properties of the model as far as $\mathcal{D}$, F and β are concerned (these are studied in section 3). The properties of γ, on the other hand, rely on the features of the decision making model of each agent (examples of such models are reviewed in section 4). Using Kakutani's theorem for an appropriate topology we have to prove that $\mathcal{D} \times F$ is compact and convex and that (β,γ) is a non-empty, compact and convex-valued correspondence and has a closed graph.

3. Fixed-Point Properties of the Model

In this part we omit all time and agent indexes unless needed. The fixed-point properties of the model

$\mathcal{D} \times F$ is compact and convex

β is continuous ,

are proved for the μ-weak topology on the set of transitions, as defined in Appendix 2.

The properties of γ depend on the definition of the way agents are making their decisions; this is not part of the model. Two examples of correspondence γ are given in section 4.

On H, compact by assumption, there is the "real" (unknown) probability μ. Then, according to Appendix 2, we transform for every $i \in I$

$$\alpha_i : H \longrightarrow A_i \quad \text{in} \quad \alpha_i \otimes \mu : \longrightarrow H \times A_i \ .$$

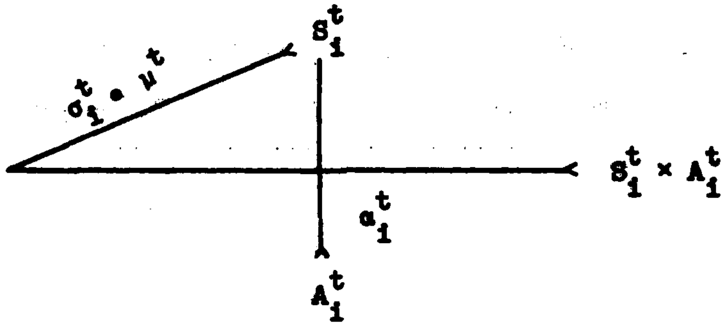
<u>Definition</u>: α_n is said to converge μ-weakly toward α, iff $\alpha_n \otimes \mu$ converges weakly toward $\alpha \otimes \mu$.

Consistent with the notation of Appendix 2, let us call $\mathcal{D}'$ the set of probabilities on $H \times A$. $\mathcal{D}'$ is compact for the topology of the weak convergence of measures on the compact set $H \times A$. $\mathcal{D}'\mu$, the set of probabilities on $H \times A$ whose marginals on H are μ, is compact too, and so is $\mathcal{D}\mu$, the product of sets of equivalence μ-a.e. classes of transitions. $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}\mu$, $\mathcal{D}'\mu$ are convex.

We then have the first result.

<u>Result 1</u>: For every probability μ on H, $\mathcal{D}\mu$ is compact convex.

On $S_i^t \times A_i^t$ the probability is the cross-product of the probability on S_i^t: $\sigma_i^t \bullet \mu^t$, and of the transition α_i^t from S_i^t to A_i^t, which is in fact, the distribution that agent i decides on the set of his actions, given his signal. We have:



Let us call

$$\nu_i^t = \alpha_i^t \otimes (\sigma_i^t \circ \mu^t) \quad .$$

If F_i is the set of transitions from $S_i^t \times A_i^t$ to S_i^{t+1} (we now write $S_i \times A_i \longrightarrow S_i^{+1}$ and drop the index t), let F_i' be the set of probabilities on $S_i \times A_i \times S_i^{+1}$; $F_i'\nu_i$ the set of probabilities on $S_i \times A_i \times S_i^{+1}$ whose marginals on $S_i \times A_i$ are ν_i. For the ν_i-weak topology on $F_i'\nu_i$, all these sets are compact. F_i, F_i', $F_i\nu_i$, and $F_i'\nu_i$ are convex. Then $F = \mathop{X}_{i \in I} F_i\nu_i$ is compact and convex being a finite product of compact convex sets for the topology canonically defined by $\Psi^n \to \Psi \iff \forall i \in I \;\; \Psi_i^n \to \Psi_i \;\; \nu_i$-weakly.

<u>Result 2</u>: For every probability μ, and family of decisions α^t, F is a compact set for the above topology.

To prove the continuity of β, we translate the problem in terms of transition to a problem in terms of measures:

$$\alpha_n \in \mathcal{D}\mu \longrightarrow (\alpha_n \circ \sigma) \otimes \mu \in \mathcal{D}'\mu$$

$$\beta_i(\alpha_n) \in F_i \longrightarrow \beta(\alpha_n) \otimes [(\alpha_{i,n} \otimes \sigma_i) \circ \mu] \in F_i\nu_i \quad .$$

and if $\alpha_n \xrightarrow[\mu-w]{} \alpha$ which means that $(\alpha_n \otimes \sigma) \otimes \mu \xrightarrow[w]{} (\alpha \otimes \sigma) \otimes \mu$ we have to prove that

$$(3.1) \qquad \beta(\alpha_n) \otimes [(\alpha_n \otimes \sigma) \circ \mu] \xrightarrow[w]{} \beta(\alpha) \otimes [(\alpha \otimes \sigma) \circ \mu]$$

Recall that $\beta(\alpha_n) = \sigma^+ \circ Q(\alpha_n) \circ M(\alpha_n)$ with $M(\alpha_n)$ verifying the defining equality $M(\alpha_n) \otimes (\alpha_n \otimes \sigma) \circ \mu = (\alpha_n \otimes \sigma) \otimes \mu$.

(3.1) means

$$\forall f \in \mathcal{B}_1(A \times S \times H^+) \quad ,$$

$$(3.2) \qquad \int_{A\times S\times H^+} f(a,s,h^+)\beta(\alpha_n)(a,s,dh^+)[(\alpha_n \otimes \sigma) \circ \mu](da,ds) \quad \text{converges.}$$

(3.2) is

$$\int_{A\times S\times H^+} f(a,s,h^+) \int_H Q(\alpha_n)(\sigma^+(h),dh^+)M(\alpha_n)(a,s,dh)[(\alpha_n \otimes \sigma) \circ \mu](da,ds)$$

By the definition of $M(\alpha_n)$, this implies $\forall g \in \mathcal{B}_1(A \times S)$,

$$\int_H \int_{A\times S} g(a,s)M(\alpha_n)(a,s,dh)[(\alpha_n \otimes \sigma) \circ \mu](da,ds)$$

$$= \int_H \mu(dh) \int_{A\times S} g(a,s)(\alpha_n \otimes \sigma)(h,da,ds) \ .$$

So (3.2) becomes, since $\forall h^+, f(a,s,h^+) = g(a,s)$

$$(3.3) \qquad \int_H \mu(dh) \int_{A\times S\times H^+} f(a,s,h^+)Q(\alpha_n)(\sigma^+(h),h^+)(\alpha_n \otimes \sigma)(da,ds) \ .$$

We want to prove that (3.3) converges toward

$$\int_{A\times S\times H^+} f(a,s,h^+)\beta(\alpha)(a,s,dh^+)[(\alpha \bullet \sigma) \circ \mu](da,ds)$$

which becomes under the same simplifications:

$$\int_H \mu(dh) \int_{A\times S\times H^+} f(a,s,h^+)Q(\alpha)(\sigma^+(h),dh^+)(\alpha \bullet \sigma)(h,da,ds) \ .$$

We are thus confronted with proving the following:

$$\int_H \mu(dh) \int_{K\times L} f(k,\ell)Q(\nu_n)(h,d\ell)\nu_n(h,dk)$$

$$- \int_H \mu(dh) \int_{K\times L} f(k,\ell)Q(\nu)(h,d\ell)\nu(h,dk) \quad \text{converges toward} \quad 0 \ .$$

This can be shown by adding and subtracting the term

$\int_H \mu(dh) \int_{K\times L} f(k,\ell)Q(\nu)(h,d\ell)\nu_n(h,dk)$. We obtain

$$\int_H \mu(dh)\Bigg\{\Bigg[\int_{K\times L} f(k,\ell)Q(\nu_n)(h,d\ell)\nu_n(h,dk) - \int_{K\times L} f(k,\ell)Q(\nu)(h,d\ell)\nu_n(h,dk)\Bigg]$$

$$+ \Bigg[\int_{K\times L} f(k,\ell)Q(\nu)(h,d\ell)\nu_n(h,d\ell) - \int_{K\times L} f(k,\ell)Q(\nu)(h,d\ell)\nu(h,d\ell)\Bigg]\Bigg\} \ .$$

The second bracket is:

$$\int_K \nu_n(h,d\ell) \int_L f(k,\ell)Q(\nu)(h,d\ell) - \int_K \nu(h,d\ell) \int_L f(k,\ell)Q(\nu)(h,d\ell) \ .$$

If we call $g(h,\ell) = \int_L \delta(k,\ell)Q(\nu)(h,d\ell)$, $g \in \beta_1(K \times L)$, we see

$$\nu_n \xrightarrow[\mu\text{-}w]{} \nu \Rightarrow \int_H \mu(dh) \ [\text{second bracket}\ ] \longrightarrow 0 \quad .$$

The first bracket is:

$$\int_L Q(\nu_n)(h,d\ell) \int_K \delta(k,\ell)\nu_n(h,dk) - \int_L Q(\nu)(h,d\ell) \int_K \delta(k,\ell)\nu_n(h,dk)$$

and if we call $g_n(h,k) = \int_K \delta(k,\ell)\nu_n(h,dk) \in \beta_1(H \times K)$, we have

$$\int_H \mu(dh) \int_L g_n(h,k)Q(\nu_n)(h,d\ell) - \int_L g_n(h,k)Q(\nu)(h,d\ell) \quad .$$

We shall need the following

<u>Lemma</u>: Q is a continuous function of α: $\alpha_n \xrightarrow[\mu\text{-}w]{} \alpha \Rightarrow Q(\alpha_n) \xrightarrow[\mu\text{-}w]{} Q(\alpha)$. Recall that this α is the aggregate of α_i and α: $H \multimap A = \underset{i \in I}{\times} A_i$.

Hence, for every $g \in \beta_1(H \times K)$:

$$\forall\ \varepsilon\ \exists\ N\ \forall\ n > N\varepsilon \quad \left| \int_H \mu(dh) \left[\int_L g(h,k)Q(\nu_n)(h,d\ell) - \int_L g(h,k)Q(\nu)(h,d\ell) \right] \right| \leq \varepsilon$$

in particular, for $n > N\varepsilon$ we have the same result for g_n instead of g, thus proving the convergence toward 0 of the first bracket. This ends the proof of the following result

<u>Result 3</u>: β is a continuous function from $\mathcal{D}$ to F for the μ-weak topology.

We still have to prove the Lemma, which is by itself an interesting result concerning the process Q which is continuous in α.

Recall that $Q(\alpha_n)$ is defined by: $H^{+1} = \Omega^{+1} \times H \times A$, $\forall \bar{E} \in \mathcal{O}^{+1}$, $\forall \bar{H} \in \mathcal{H}$, $\forall \bar{A} \in \mathcal{A}$, $\forall h \in H$, $Q(\alpha_n)(\bar{h}, \bar{E} \times \bar{H} \times \bar{A}) = \chi_{\bar{H}}(h) \int_{\bar{A}} \pi(h,a,\bar{E})\alpha_n(h,da)$, or $\forall h \in H$, $\forall f \in \mathcal{B}_1(H \times H^{+1})$, with $H^{+1} = (w,h,a)$:

$$\int_H \mu(dh) \int_{H^+} f(h,h^+)Q(\alpha_n)(h,dh^+) = \int_H \mu(dh) \int_{\Omega^{+1}\times A} f(h,w,a)\pi(h,a,dw)\alpha_n(h,da) \quad .$$

As

$$\int_\Omega f(h,w,a)\pi(h,a,dw) \in \mathcal{B}_1(H \times A)$$

the last integral converges toward

$$\int_H \mu(dh) \int_{\Omega^{+1}\times A} f(h,w,a)\pi(h,a,dw)\alpha(h,da) = \int_H \mu(dh) \int_{H^{+1}} f(h,h^+)Q(\alpha)(h,dh^+)$$

when $\alpha_n \xrightarrow[\mu\text{-}w]{} \alpha$.

4. <u>Decisions in a Temporary Equilibrium</u>

A way to define the relation γ between the forecasts of agents ψ^{t+1} and their decision α^t is to suppose that some sort of equilibrium occurs at time t: agents are optimizing in a consistent way. Now we consider two different notions. First, a general Nash equilibrium where agents maximize their total expected gain, relative to their expectations on the future signals and where they take the decisions of other agents as given.

Second, a Walrasian temporary equilibrium of pure exchange markets with assets: we shall use Grandmont's model ([1970] and [1974]).

Nash Equilibrium

Definition: At each time t (that we no longer indicate), agent i makes his stationary decision α_i. Recall that α_i is a transition from S_i to A_i and a stationary decision means that agent i chooses an a_i in A_i with probability $\alpha_i(s_i)$ and that next time he plans to choose a_i^{+1} with probability $\alpha_i(s_i^{+1})$ if he receives signal s_i^{+1}, and so on: $\alpha_i^{\tau}(s_i) = \alpha_i(s_i)$. If his utility function at time t depends on his decision and on other agents' actions $(a_j)_{j\neq i}$, let us write $U_i((a_j)_{j\neq i},\alpha_i)$ for the expected utility. For example, we could have

$$U_i((a_j)_{j\neq i},\alpha_i) = \int_{A_i^+} U_i \quad (a_j)_{j\neq i},a_i^+) \prod_{\tau=0}^{T} \alpha_i(s_i^{\tau},da_i^{\tau})$$

where T is a finite horizon. The product of the $\alpha_i(s_i)$ means that the actions at time t are independent of the actions made at other times.

We assumed also that decisions of agents are independent, so that the distribution on $A = \underset{i\in I}{\times} A_i$ is $\prod_{i\in I} \alpha_i$.

In a Nash equilibrium agent i maximizes his utility taking the decisions of other agents as given, so that his payoff would be:

$$\int_A U_i((a_j)_{j\neq i}, a_i(s_i^+)) \prod_{i\in I} \alpha_i(s_i, da_i)$$

where $\alpha_j(s_j)$ is given for $j \neq i$.

This expression depends on the future signals s_i^+. Agent i has a forecast on this, Ψ_i^{+1}: $S_i \times A_i \longrightarrow S_i^{+1^+}$. In order to avoid the complexity due to the product $S_i \times A_i$, we simplify Ψ_i^{+1} as a transition from S_i to $S_i^{+1^+}$. Then the total expected payoff relative to the expectation Ψ_i^{t+1} on the future signals s_i^+ of agent i, given the others' decisions, is:

$$G_i(\alpha, \Psi_i(s_i)) = \int_{S_i^{+1}} \Psi(s_i, ds_i^{+1^+}) \int_A U_i((a_j)_{j\neq i}, a_i(s_i^+))$$

$$\cdot \prod_{i\in I} \alpha_i(s_i, da_i) \quad .$$

A Nash equilibrium is an $\alpha^* = (\alpha_i^*)_{i\in I}$ such that:

$$(4.1) \qquad (\forall\, i \in I), (\forall\, \alpha_i \in \mathcal{D}_i), G_i((\alpha_j)_{j\neq i}, \alpha_i^*, \Psi_i) \geq G_i(\alpha, \Psi_i) \quad .$$

So our correspondence γ: $\mathcal{F} \longrightarrow D$ which relates forecasts of agents to their decisions in a Nash equilibrium is defined by:

$$\gamma(\Psi) = \{\alpha^* \mid \forall\, i \ (4.1) \text{ holds}\} \quad .$$

Properties of γ.

(1) γ is non-empty. This is because, with our hypothesis, a Nash equilibrium exists; we verify that the Nash theorem holds:

(a) $\mathcal{D}_i$ is convex and compact,

(b) G_i is continuous on $\underset{i\in I}{\times} \mathcal{D}_i$

(c) $\forall\, \alpha, \{\alpha_i' \,|\, G_i((\alpha_j)_{j\neq i}, \alpha_i')$ is maximum$\}$ is convex.

Proof:

(a) has already been established for the μ-weak topology.

(b) let $\alpha_n \xrightarrow[\mu\text{-}w]{} \alpha$, which means that $\forall\, i \in I, \alpha_{i,n} \xrightarrow[\sigma_i\circ\mu\text{-}w]{} \alpha_i$, or $\forall\, i \in I, \alpha_{i,n} \circ \sigma_i \xrightarrow[\mu\text{-}w]{} \alpha_i \circ \sigma_i$. As we often did, let us write α_i for $\alpha_i \circ \sigma_i$.

Lemma: $\forall\, i \in I, \alpha_{i,n} \xrightarrow[\mu\text{-}w]{} \alpha_i \Rightarrow \prod_{i\in I} \alpha_{i,n} \xrightarrow[\mu\text{-}w]{} \prod_{i\in I} \alpha_i$.

Proof: It is sufficient to prove it for $I = \{1,2\}$:

We have $\forall\, f_i \in \mathcal{B}(A_i \times H), \int_H d\mu \int_{A_i} f_i \, d\alpha_{i,n} \longrightarrow \int_H d\mu \int_{A_i} f_i \, d\alpha_i$.

We need $\forall\, f_i \in \mathcal{B}(A \times H), \int_H d\mu \int_A f \, d\Pi\alpha_{i,n} \longrightarrow \int_H d\mu \int_A f \, d\Pi\alpha_i$.

But we can decompose $\int d\mu \int f \, d\alpha_{1_n} d\alpha_{2_n} - \int d\mu \int f \, d\alpha_1 d\alpha_2$ into

$$\int d\mu \int_{A_1\times A_2} f(a_1,a_2,h)\alpha_{1_n}(h,da_1)\, \alpha_{2_n}(h,da_2) - \int d\mu \int f d\alpha_{1_n} d\alpha_2$$

$$+ \int d\mu \int f \, d\alpha_{1_n} d\alpha_2 - \int d\mu \int f \, d\alpha_1 d\alpha_2 .$$

$$\int_{A_1} f(a_1,a_2,h)\alpha_{1_n}(h,da) \in \mathcal{B}_1(A_2 \times H) \quad ,$$

thus the first difference converges toward 0, and as

$$\int_{A_2} f(a_1,a_2,h)\alpha_2(h,da) \in \mathcal{B}_1(A_1 \times H) \quad ,$$

the second difference also converges toward 0.

Now $G_i(\alpha_n,\Psi_i(s_i)) = \int_{S^+} \int_A U_i \prod_{i\in I} d\alpha_{i,n} d\Psi_i(s_i)$.

We want to prove that if $\alpha_n \xrightarrow[\mu-w]{} \alpha$, $G_i(\alpha_n(s_i),\Psi_i) \longrightarrow G_i(\alpha(s_i),\Psi_i)$

$\sigma \circ \mu$-a.e. but $\int_{S^+} U_i d\Psi_i(s_i)$ is bounded by 1 (if U_i was not, take $1 - 1/U_i$). So if $\Pi\alpha_{i,n} \longrightarrow \Pi\alpha_i$, $G_i(\alpha_n(s_i),\Psi_i) \longrightarrow G_i(\alpha(s_i),\Psi_i)$ $\sigma \circ \mu$-a.e.

(c) Let α_i' and α_i'' be such that $G_i(\alpha)$ is maximum in α_i and call $M = G_i(\prod_{j\neq i} \alpha_j \times \alpha_i') = G_i(\prod_{j\neq i} \alpha_j \times \alpha_i'')$. Then for any $\lambda \in (0,1)$ as $G_i(\alpha)$ is linear in α_i:

$$G_i(\prod_{j\neq i} \alpha_j \times (\lambda\alpha_i' + (1-\lambda)\alpha_i'')) = \lambda M + (1-\lambda)M = M \quad .$$

Thus $\lambda\alpha_i' + (1-\lambda)\alpha_i''$ is such that $G_i(\alpha)$ is maximum in α_i and the set of maxima is convex.

We have thus proved the first property that γ is non-empty.

(2) <u>γ is convex valued</u> for every $\Psi \in P$, and for every α' and α'' in $\lambda(\Psi)$:

$$\left.\begin{aligned} G_i(\prod_{j\neq i} \alpha_j \times \alpha_i') &\geq G_i(\alpha) \\ G_i(\prod_{j\neq i} \alpha_j \times \alpha_i'') &\geq G_i(\alpha) \end{aligned}\right\} \quad \forall \alpha \quad , \quad \forall i$$

Thus multiplying the first line by $\lambda \in (0,1)$ and the second by $1-\lambda$, adding the two lines we have:

$$\lambda G_i(\Pi\alpha_j \times \alpha_i') + (1 - \lambda)G_i(\Pi\alpha_j \times \alpha_i'') \geq G_i(\alpha)$$

which, as G_i is linear in α_i, proves that $\lambda\alpha_i' + (1 - \lambda)\alpha_i''$ is in $\gamma(\Psi)$.

(3) <u>γ is compact valued</u>. It is sufficient to prove that $\gamma(\Psi)$ is closed as F is compact.

If α_n is a sequence of strategies belonging to $\gamma(\Psi)$, and if $\alpha_n \longrightarrow \alpha$, as G_i is continuous in α, $G_i(\alpha_n) \longrightarrow G_i(\alpha)$ and thus $G_i(\alpha)$ is maximum and α is an equilibrium for Ψ.

(4) <u>γ has a closed graph</u>. Let (α_n,Ψ_n) be a sequence of $\mathcal{D} \times F$ converging toward $(\bar{\alpha},\bar{\Psi})$ so that $\forall n\ \alpha_n \in \gamma(\Psi_n)$. γ has a closed graph means that $\bar{\alpha} \in \gamma(\bar{\Psi})$.

$\alpha_n \in \gamma(\Psi_n)$ means that

$$\forall i \in I, \forall \alpha_i', \ G_i(\alpha_n,\Psi_{i,n}) \geq G_i((\alpha_{j_n})_{j\neq i},\alpha_i',\Psi_{i,n}) \quad .$$

We prove now that

$$G_i(\alpha_n,\Psi_{i,n}) \longrightarrow G_i(\bar{\alpha},\bar{\Psi}_i)$$

and

$$\forall \alpha_i', \ G_i((\alpha_{j_n})_{j\neq i},\alpha_i',\Psi_{i,n}) \longrightarrow G_i((\bar{\alpha}_j)_{j\neq i},\alpha_i',\bar{\Psi}_i) \quad ,$$

so that

$$\forall i \ \ \forall \alpha_i, \ G(\bar{\alpha},\bar{\Psi}) \geq G_i((\bar{\alpha}_j)_{j\neq i},\alpha_i',\bar{\Psi}_i) \quad ,$$

or $\bar{\alpha} \in \gamma(\bar{\Psi})$.

From the lemma of property 1(b), we have

<u>Lemma 1</u>: If $\forall\, i \in I,\ \alpha_{i,n} \xrightarrow[\mu-w]{} \alpha_i,\ \prod_{i\in I} \alpha_{i,n} \longrightarrow \prod_{i\in I} \alpha_i$.

<u>Lemma 2</u>: $G_i(\alpha_n, \Psi_{i,n}) - G_i(\bar{\alpha}, \bar{\Psi}_i) \longrightarrow 0$.

<u>Proof</u>: The difference can be written

$$\int_{S^+} d\Psi_{i,n} \int_A U_i \; d\alpha_n - \int_{S^+} d\Psi_{i,n} \int_A U_i \; d\bar{\alpha} + \int_{S^+} d\Psi_{i,n} \int_A U_i \; d\bar{\alpha}$$

$$- \int_{S^+} d\bar{\Psi}_i \int_A U_i \; d\bar{\alpha} \quad .$$

From Lemma 1, $\forall\, i\ \alpha_{i,n} \longrightarrow \bar{\alpha}_i,\ \alpha_n = \prod_{i\in I} \alpha_{i,n} \longrightarrow \bar{\alpha} = \prod_{i\in I} \alpha.$

Thus, in the first difference, $\int_A U_i \; d\alpha_n - \int_A U_i \; d\bar{\alpha}$ can be majored by ε as soon as $n > N\varepsilon$, for any $\varepsilon > 0$. As $\Psi_{i,n} \longrightarrow \bar{\Psi}_i$, and $\int U_i \; d\bar{\alpha} \in \mathcal{B}_1(S^+)$, the last difference converges toward 0.

<u>Lemma 3</u>: $\forall\ \alpha_i',\ G_i((\alpha_{j,n})_{j\neq i}, \alpha_i', \Psi_{i,n}) \longrightarrow G_i((\bar{\alpha}_j)_{j\neq i}, \alpha_i', \Psi_{i,n})$.

<u>Proof</u>: This is obviously a particular case of Lemma 2, when the sequence $\alpha_{1,n} \ldots \alpha_{i,n} \ldots \alpha_{I,n}$ is replaced by $\alpha_{1,n} \ldots \alpha_n' \ldots \alpha_{I,n}$, which converges toward $\bar{\alpha}_1 \ldots \alpha_i' \ldots \bar{\alpha}_I$.

We have therefore proved the sufficient conditions for γ:

1. γ is non-empty.
2. γ is convex valued.

3. γ is compact valued.

4. γ has a closed graph.

Together with the properties of the model, β is continuous and the sets $\mathcal{D}$ and $\mathcal{F}$ are compact and convex, hence we can state:

<u>Theorem</u>: In this model, when γ is defined by a temporary Nash equilibrium, the correspondence (β,γ) has a fixed-point.

Or, less formally: In this process of decision making, if agents make their decision at each time according to a Nash equilibrium, then there exists an informational equilibrium for the process.

This result comes partly from the general properties of the model (properties of $\mathcal{D}$, $\mathcal{F}$ and β) and from the fact that the way that actions of agents are defined is a "sufficiently continuous" function of their forecasts in the temporary Nash equilibrium (properties of γ).

<u>Walrasian Equilibrium</u>

In a sequence of pure exchange markets where endowments are random (they are given by an unknown random process) and where agents trade consumption goods at time t and make plans for consumptions at time $t + 1$, an equilibrium at time t can occur. This is the temporary Walrasian equilibrium as studied by Grandmont, among others.

In his model, time has 2 values. (This could mean that the random process of endowments is a homogeneous Markov process with constant kernel, so that the economy is repeated identically.)

Actions of agents at time 1 are consumptions: A_i^1 is a compact set of $\mathbb{R}_+^{\ell}$ and a consumption plan for next period A_i^2 which is a compact set of $\mathbb{R}_+^{\ell}$, and $\alpha_i^1 = (a_i, \alpha_i^2)$ where α_i^1 is a function from the set of signals. The link between the two periods is an asset (called money) which can be carried forward from date 1 to date 2. Let $a_i^{\ell} = m_i$ be the quantity of money that agent i carries forward. The signals are taken to be the equilibrium vector of prices which is a function of the random endowments w^t through the maximization of expected utility relative to the forecast Ψ_i^2: $S^1 \longrightarrow S^2$. α_i^1 is chosen to maximize:

$$V_i(s^1) = \int_{S^2} U_i(a_i^1(s^1), \alpha_i^2(s^1, s^2))\Psi(s^1, ds^2)$$

under the budget constraints below. Set $a_i^1 = (x_i^1, m_i^1)$ where m_i is money, the only transferable good which is totally consumed in period 2.

$$\begin{cases} s^1 x_i^1 + m_i^1 \leq s^1 w^1 + m_i^0 \\ \\ \int_{S^2} s^2 x_i^2 \Psi(s^o, ds^2) \leq \int_{S^2} s^2 w_i^2 \Psi(s^1, ds^2) + m_i^1 \end{cases} .$$

If α_i^1 is the demand function of agent i; $\alpha_i^1(s^1) = (x_i^1, m_i^1)$, it must then follow Walrasian Law:

$$\sum_{i \in I} \alpha_i(s^1) = \sum_{i \in I} (\alpha_i^1 + m_i^0).$$

The necessary condition for existence of an equilibrium $(\bar{s}^1, \bar{\alpha})$ (Radner, [1980]), that is (namely) the excess demand correspondence is an upper semi-continuous non-empty convex valued correspondence ζ, such that $\forall \nu \in \zeta(s)$, $s\,\nu = 0$, was proved by Grandmont [1970] under the following assumptions on agents

--Initial endowments are positive and the initial endowment of money is non-zero.

--Utility functions are continuous concave, strictly increasing and time separable

$$U_i(a_i^1(s^1), \alpha_i^2(s^1,s^2)) = U_i^1(a_i^1(s^1)) + U_i^2(\alpha_i^2(s^1,s^2)) \cdot$$

and on the forecast functions:

--Ψ_i is such that if s^1 is a strictly positive price, $\Psi_i(s^1)$ gives a positive probability to every set of strictly positive prices for period 2.

--Ψ_i is weakly continuous.

--Ψ_i is tight: $\forall\, \varepsilon > 0$, $\exists\, K$ compact $\subset S^2$, $\forall\, s^1 \in S^1$, $\Psi(s^1,K) > 1 - \varepsilon$ (our hypothesis that S is compact implies this last hypothesis).

Then, and for this class of forecast functions, a temporary equilibrium $(\bar{s}^1, \bar{\alpha})$ exists.

<u>Properties of γ.</u>

1. $\gamma(\Psi)$ is non-empty.

2. γ is convex valued as if α' and α'' maximize $\int U_i(\alpha)d\Psi$,

$$\int U_i(\lambda\alpha' + (1 - \lambda)\alpha'')d\Psi \geq \int \lambda U_i(\alpha')d\Psi + \int (1 - \lambda)U_i(\alpha'')d\Psi \quad ,$$

as U_i is concave, and for every α,

$$\int U_i(\lambda\alpha' + (1 - \lambda)\alpha'')d\Psi \geq \lambda \int U_i(\alpha)d\Psi + (1 - \lambda)\int U_i(\alpha)d\Psi \geq \int U_i(\alpha)d\Psi \quad ,$$

as α' and α'' are optimal. So $\lambda\alpha' + (1 - \lambda)\alpha''$ is optimal too.

3. γ is compact valued as $\gamma(\Psi)$ is closed and this follows from U_i being continuous.

4. γ has a closed graph: if $(\alpha_n, \Psi_n) \xrightarrow[n]{} (\bar{\alpha}, \bar{\Psi})$ so that $\forall n \in \mathbb{N}$, $\alpha_n = \gamma(\Psi_n)$. Let us prove that $\bar{\alpha} \in \gamma(\bar{\Psi})$.

$$\forall \alpha, \ \int U(\alpha_n)d\Psi_n \geq \int U(\alpha)d\Psi_n$$

$$\forall \alpha, \ \lim_n \int U(\alpha_n)d\Psi_n \geq \int U(\alpha)d\Psi \quad \text{as } U \text{ is bounded, and}$$

$$\lim_n \int U(\alpha_n)\, d\Psi_n = \int U(\alpha_n) - U(\bar{\alpha})d\Psi_n + \int U(\bar{\alpha})d\Psi_n \xrightarrow[n]{} \int U(\bar{\alpha})d\Psi \quad .$$

Thus, there exists an informational equilibrium in a process of markets if it satisfies Grandmont's conditions for temporary Walrasian equilibrium to exist at each time.

Conclusion: Let us point out that the temporary equilibrium concept is not essential in this model for defining the relation between the forecast and the decision $\gamma(\Psi) = \alpha$. Indeed, in Shefrin's model which gave us the seminal notion of informational equilibrium there

is no such temporary equilibrium: agents only choose an optimal policy according to a dynamic programming process which is "optimal with respect to the process in question." (The process in question is the process that we call $\beta(\alpha)$ which generates the future distribution.) The important result is that for an informational equilibrium to exist, it is sufficient that γ is continuous, loosely speaking. It is satisfying that this is true when γ is defined by a Nash or temporary Walrasian equilibrium.

The continuity of γ is sufficient, because our model describes a process in which the future distribution is a continuous function of the decision. This result, as the model itself is quite general, could be used to study the learning process of agents which would lead to a rational expectation equilibrium. A rational expectation equilibrium could then be a limit of this informational equilibrium when the process becomes stationary.

5. Conclusion

We used a structure which gives the notion of equilibrium of decisions and forecasts a precise meaning. In this process of decision making, expectations are formalized by forecast functions which agents use to make their decisions. Here the model is given and we did not try to describe the way the Law of Motion π^{t+1} was generated by the system. This way our results cannot be easily compared to existence results of rational expectations equilibrium for instance, where the main problem is the formalization of the process of price formation.

But given the existence of this Motion Law, we proved that the model had the necessary "continuous" property to allow existence of an equilibrium. We separated the process itself from the formation of decisions through forecast functions, to emphasize the role played by each specific model of decision making. If this latter process is continuous as well, then an equilibrium can exist. Now each part of this model should be implemented to relate more closely to and help solve economic decision problems.

Concerning the model itself we could think of π^{t+1} as a formulation of an equilibration function relating equilibrium decisions and prices at time t and $t + 1$.

Actually, it need not be an equilibrium model. In a disequilibrium model π^{t+1} could also be defined; it would be a more general Motion Law.

Concerning the decision making process of agents, an explicit formulation of the formation and implementation of forecasts should be developed. Then a rational expectation equilibrium could be thought of as a stationary process of informational equilibria.

Introduction to Appendices

Probability Transitions

Our main interest here is a description of the way agents take uncertainty into account and the role this plays in defining states of economies, if not equilibrium. Thus, setting aside as far as possible the problems related to the way demand functions are derived from utility functions, we shall try to focus on the forecasting, learning, and more generally, on the description of consistent agents using information to diminish the uncertainty they have to deal with. The mathematical tools used to formalize uncertainty, random events and forecasting all derive from probability and statistics theory. It seems that, although dealing mainly with the same problem, authors differ a great deal in their models, the way they treat them and the results they obtain. Feeling a need to compare and integrate different works in the same framework, we have tried to build a general model of markets under uncertainty.

Seeking generality, a tool more general than probability distributions was needed: probability transitions. They have been used very widely in dealing with conditional probability. They are also the simplest way of describing a family of distributions indexed by a parameter as is the case in most statistical models. They represent what is called in game theory a mixed strategy (an agent assigning to each element of his set of information a probability distribution over the set of pure strategies). Thus they allow us to

represent (and to conceive) concepts by the same means; agents using their information or deciding their actions.

Another reason why probability transitions are useful is that the complicated definition of composition gives birth to a very simple notion (it is the same as the composition of function, which is indeed a particular case of the composition of transition). Thus we see more easily the structure of models in which, without this tool, we were lost in long calculations not appealing to intuition.

Thus, the description of a Bayesian learning process is rather simple and easy to describe by a drawing. In dynamic processes we very often have to deal with Markov chains. I think that the use of composition and "crossed product" (an operation I define which generalizes composition and is very useful as soon as we have product spaces, as usually is the case for stochastic processes) makes clear how assumptions about the processes are related to the results (Appendix 4).

This first appendix gives definitions and algebraic properties of probability transitions (parts 1, 2, 3 and 4), then applications to Bayesian estimations (5), conditional probabilities (6) and Markov processes (7). The second appendix defines a topology on a set of transitions which is consistent with the weak topology on a set of measures. The third describes the formation of forecasts and the fourth is a reformulation of models and results on stationary equilibrium.

Appendix 1

Transition Probability as a Modelling of Relations Under Uncertainty.

Because in models describing decisions of agents we deal very often with probability transition (to each signal he receives, agents assign a probability distribution on a space) and because measurable correspondences (hence measurable functions) and probability are particular cases of probability transitions, it seems useful to emphasize the role of these morphisms. In doing so it occurred to me that an operation was needed to describe Bayesian inference and this operation appeared later to be very useful in dealing with product spaces. I first called it "produit bizarre," but "crossed product" seems to be simpler.

1. Definition of a probability transition

Let $A = (A,\mathcal{A})$ and $B = (B,\mathcal{B})$ be two measurable spaces and let $T\colon A \times \mathcal{B} \longrightarrow [0,1]$ be such that:

(i) $\forall a \in A$, $T(a,\cdot)\colon \mathcal{B} \longrightarrow [0,1]$ is a probability on $(B,\mathcal{B})$.

(ii) $\forall B_0 \in \mathcal{B}$, $T(\cdot,B_0)\colon A \longrightarrow [0,1]$ is measurable for the Borel sigma-algebra on $[0,1]$.

Then T is called a probability transition, or shorter, a transition from A to B, and we shall denote it $T\colon A \longrightarrow\!\!\!< B$.

2. Particular cases of transition.

(i) Probability: A probability P on $(A,\mathcal{A})$ can be viewed as a transition from any space of only one element $(\cdot,\{\{\cdot\},\phi\})$ to A,

such that $A_0 \in \mathcal{A}$ $P(\cdot, A_0) = P(A_0)$. So we shall denote $P: \cdot$ —< A for a probability on the measurable space A.

(ii) <u>Measurable function</u> $f: A \longrightarrow B$. We can associate to f the unique transition $F: A$ —< B such that

$$\forall a \in A \; \forall B_0 \in \mathcal{B}, \quad F(a, B_0) = \begin{cases} 1 \text{ if } f(a) \in B_0 \\ 0 \text{ if not} \end{cases} .$$

3. <u>Composition of transition</u>.

<u>Definition</u>: We call composition of transitions T from B to C and S from A to B, and we write $T \circ S$, the transition from A to C defined by:

$$(\forall a \in A)(\forall C_0 \in \mathcal{C}) \quad T \circ S(a, C_0) = \int_B T(b, C_0)\, S(a, db) \quad .$$

(Let us be reminded that if f is an integrable function on B, and if S is a transition from a to B, for each a, $S(a, \cdot)$ is a probability on B and we write $\int_{B_0} f(b)\, S(a, db)$ for the integral of f relative to the measure $S(a, \cdot)$.)

(i) <u>Example</u>: Composition of functions:

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & C \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dr, "G"] & \\ A \arrow[ur, "F"] \arrow[rr, "G \circ"'] & & C \end{tikzcd}$$

$$(\forall\, a \in A)(\forall\, C_0 \in \mathcal{C}) \quad G \circ F(a,C_0) = \int_B G(b,C_0)F(a,db)$$

$$= \int_B 1_{g^{-1}(C_0)}(b)F(a,db)$$

$$= F(a,g^{-1}(C_0)) = \begin{cases} 1 & \text{if } \ f(a) \in g^{-1}(C_0) \\ 0 & \text{if not.} \end{cases}$$

And the transition associated with $g \circ f$ takes for a and C_0,

the value $\begin{cases} 1 & \text{if } \ g \circ f(a) \in C_0 \\ 0 & \text{if not} \end{cases}$ which is $G \circ F(a,C_0)$.

(ii) Induced transition: Let T be a transition from A to B and f a measurable function from B to C. If F is the transition associated to f, we can write $F \circ T$ for the composition. It is

$$\forall\, C_0 \in C,\ F \circ T(\cdot,C_0) = \int_B F(b,C_0)T(a,db) = \int_B 1_{f^{-1}(C_0)}(b)\,T(a,db) = T(a,f^{-1}(B_0)$$

which is the image of T by f.

From now on, aggregating f and F we shall write $f \circ T$ for the image of T by f.

A particular case is when P is a probability on B $(A = \cdot)$ and f a measurable function; $f \circ P$ is defined by, and $f \circ P$ is the probability induced by f.

4. Crossed product of transition:

Composition is a particular case of the following. Let S be a transition from A to B and T from B to C. We can define a transition from A to the Cartesian product $B \times C$ by defining it on

the rectangles of $\mathcal{B} \otimes \mathcal{C}$, $\forall\, a \in A$, $\forall\, B_0 \times C_0 \in \mathcal{B} \otimes \mathcal{C}$

$$\int_{B_0} T(b, C_0) S(a, db)$$

This defines a transition from A to $B \times C$ as we shall show below. Let us write $T \otimes S$ for this transition, we name it crossed product of T and S and we can represent it on a diagram like this:

$$\begin{tikzcd}
 & B \arrow[d, "T"] & \\
A \arrow[ur, "S"] \arrow[rr, "T \otimes S"] & {} & B \times C \\
 & C &
\end{tikzcd}$$

<u>$T \otimes S$ is a transition</u>:

$$\forall\, a \in A,\ \forall\, B_0 \times C_0 \in \mathcal{B} \otimes \mathcal{C},\ T \circ S(a, B_0 \times C_0) = \int_{B_0} T(b, C_0) S(a, db) \ .$$

a) $\forall\, E \in \mathcal{B} \otimes \mathcal{C}$, $\forall\, a \in A$, $T \otimes S(a, E)$ is defined as the unique prolongment of $T \otimes S(a, \cdot)$ which is defined as additive and monotonic (thus sigma-additive) on the sigma-algebra of rectangles of the form $B_0 \times C_0$ which generates $\mathcal{B} \otimes \mathcal{C}$ (see the Neveu theorem of prolongment, Proposition I-C-1).

b) $\forall\, E \in \mathcal{B} \otimes \mathcal{C}$, $T \otimes S(\cdot, E)$ is measurable. Because of the preceding theorem it suffices to prove it on the rectangle $B_0 \times C_0$. $T(\cdot, C_0)$ is measurable as T is a transition, thus $T(\cdot, C_0)$ is uniform limit of linear combinations of characteristic functions. To show that $T \otimes S(\cdot, B_0 \times C_0) = \int_{B_0} T(b, C_0)\, S(\cdot, db)$ is measurable, it

is sufficient to prove that $(\forall B_1 \in \mathcal{B}) \int_{B_0} 1_{B_1}(b)\, S(\cdot,db)$ is measurable, and this integral is $S(\cdot,B_0 \cap B_1)$ which is measurable as S is a transition. We can thus set the following

<u>Definition:</u> The unique transition defined from S and T by $\forall a \in A$, $\forall B_0 \times C_0 \in \mathcal{B} \otimes \mathcal{C}$, $T \otimes S(a, B_0 \times C_0) = \int_{B_0} T(b,C_0)\, S(a,db)$ is called the crossed product transition of T and S.

<u>Properties:</u> If σ_B and σ_C are the projections from $B \times C$ on B and on C, we have (aggregating the projections and their associated transitions):

$$
\begin{tikzcd}
 & B \arrow[dd, no head] & \\
A \arrow[ur, "S"] \arrow[rr, dashed, "T \otimes S"] \arrow[dr, dashed, "T \circ S"'] & & B \times C \arrow[ul, "\sigma_B"'] \arrow[dl, "\sigma_C"] \\
 & C &
\end{tikzcd}
$$

$$\sigma_B \circ [T \otimes S] = S \quad , \quad \forall B_0 \in \mathcal{B}$$

$$\sigma_B \circ [T \otimes S](\cdot,B_0) = T \otimes S(\cdot,\, B_0 \times C) = \int_{B_0} T(b,C)\, S(\cdot,db) = S(B_0)$$

$$\sigma_C \circ [T \otimes S] = T \circ S \quad , \quad \forall C_0 \in \mathcal{C}$$

$$\sigma_C \circ [T \otimes S](\cdot,C_0) = T \otimes S(\cdot,\, B \times C_0) = \int_{B_0} T(b,C_0)\, S(\cdot,db) = T \circ S(\cdot,C_0)\ .$$

Although certain care must be taken for left associativity, cross product is associative (see my thesis, Annex 1).

Particular cases of cross product:

a) Measurable functions:

$$\begin{tikzcd} & B \arrow[dd, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "(f, g \circ f)"] & & B \times C \\ & C & \end{tikzcd}$$

The measurable function associated with the crossed product transition of the associated F and G to f and g, is the couple $(f, g \circ f)$.

b) Product probability:

$$\begin{tikzcd} & B \arrow[dd, "T"] & \\ \cdot \arrow[ur, "P"] \arrow[rr, "T \otimes P = \pi"] & & B \times C \arrow[ul, "\sigma_B"'] \\ & C & \end{tikzcd}$$

Let $\pi = T \otimes P$ which is a probability on $B \times C$ then $\sigma_B \circ \pi = P$ (Neveu Proposition III, 2.1).

$$\begin{tikzcd} & B \arrow[dd, "\pi^{\sigma_B}"] & \\ \cdot \arrow[ur, "P"] \arrow[rr, "\pi"] & & B \times C \arrow[ul, "\sigma_B"'] \\ & C & \end{tikzcd}$$

Reciprocally, if B and C are closed in R^n, and if π is a probability of $B \times C$, there exists a transition from B to C called the probability conditioned by σ_B; let us write it π^{σ_B} (or $\pi(\cdot|\sigma_B)$), which is such that $\pi = \pi^{\sigma_B} \otimes P$ (or, $\pi(\cdot|\sigma_B) \otimes P$) (Raoult [1975]).

c) <u>Induced distribution</u>: In the following diagram we see that

$$\begin{tikzcd} & B \arrow[d, "\delta"] & \\ \bullet \arrow[ur, "P"] \arrow[rr, "\delta \otimes P"] & C & B \times C \end{tikzcd} \qquad \delta \otimes P(\cdot, B_0 \times C_0) = P[B_0 \cap \delta^{-1}(C_1)] \quad .$$

5. <u>Application of transitions to Bayesian estimation</u>

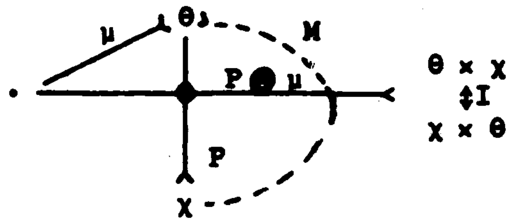
Let Θ be the set of parameters (together with a sigma-algebra, if the model doesn't imply any, let it be the set of all subsets of Θ), and let μ be a prior probability on Θ. If χ is the measurable space of experiments, let P be a transition from Θ to χ describing the statistical structure.

A Bayesian statistician will use $\pi = P \circ \mu$ as a predictable probability on X. On $\Theta \times X$ we can define $P \otimes \mu$.

Let I: $\Theta \times X \longrightarrow X \times \Theta$

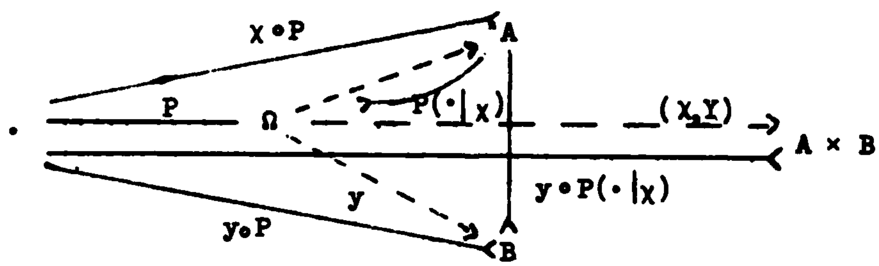
such that $(\theta, x) \longrightarrow (x, \theta)$.

If Θ and X are closed subsets of some $\mathbb{R}^n$ there exists a transition M from X to Θ such that: $I \circ (M \otimes \pi) = P \otimes \mu$.



$\forall x \in X$, $M(x,\cdot)$ is called a posterior distribution on Θ. This is the probability that the Bayesian statistician uses instead of μ, when he has observed x. He will then predict that an event $X_0 \in X$ will have the probability $P \circ M(x,X_0)$.

6. Application to Conditional Probabilities



Unhappily, the common way of writing conditional probability is the reverse of the way we wrote the transition. The probability conditional to χ on B is a transition from A to Ω such that:

$$\forall A_0 \in A,\ \forall B_0 \in B,\ \int_{A_0} P(B_0|\chi = x)[\chi \circ P](dx) = P(\chi^{-1}(A_0) \cap B_0)$$

Then Y induces a new probability on B: $Y \circ P(\cdot|\chi)$.

From the knowledge of $\chi \circ P$ and of $P(\cdot|\chi)$ one can define a "natural" probability on the product $A \times B$ by:

$$\forall A_0 \in \mathcal{A},\ \forall B_0 \in \mathcal{B},\ [Y \circ P(\cdot|\chi)] \otimes [\chi \circ P](A_0 \times B_0)$$

$$= \int_{A_0} Y \circ P(B_0|\chi = x)(\chi \circ P)(dx) \ .$$

Symmetrically, in conditioning relative to Y, we have on $B \times A$ the probability

$$\forall B_0 \in \mathcal{B},\ \forall A_0 \in \mathcal{A},\ [\chi \circ P(\cdot|Y)] \otimes [Y \circ P](B_0 \times A_0)$$

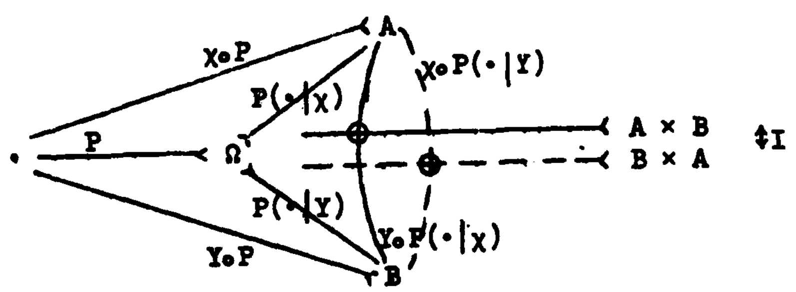
$$= \int_{B_0} \chi \circ P(A_0|Y = y)(Y \circ P)(db) \ .$$

Now if

$$I\colon\ A \times B \longrightarrow B \times A$$
$$(a,b) \longrightarrow (b,a) \quad ,$$

$$I \circ ([Y \circ P(\cdot|\chi)] \otimes [\chi \circ P]) = [\chi \circ P(\cdot|Y)] \otimes Y \circ P \quad ,$$

which is easier to look at on the diagram than to read!



7. Application to Markov process

If time is a discrete number $t \in \mathbb{N}$, a stochastic process $X = (X_t)_{t\in\mathbb{N}}$ where X_t is a random variable from (Ω, F, P) to $(S, \mathcal{S})$, then (S, X_t) is said to be Markovian if the probability induced on $(S, \mathcal{S})$ by X_t depends only on X_{t-1}, or,

$$\forall\ (x_1, \ldots, x_{t-1}) \in S_t$$

$$P(X_t \in A_t \,|\, \textstyle\bigcap_{s<t} [X_s = x_s]) = P(X_t \in A_t \,|\, X_{t-1} = x_{t-1})$$

Hence, for any measurable and bounded function f on S, if we write $(X_s)_{s\leq t} \circ P$ for the joint distribution of the X_s, $s \leq t$, and

$$X_t \circ P(\cdot \,|\, (X_s)_{s<t} = (x_s)_{s<t})$$

for the distribution of X_t conditional on all previous realizations of the X_s, we have:

$$\begin{aligned}\int f(x)[(X_s)_{s\leq t} \circ P](dx) &= \int f(x)[X_t \circ P](dx_t \,|\, (X_s)_{s<t}) \\ &= \int f(X_s)_{s<t} [(X_s)_{s<t} \circ P](dx_1, \ldots, dx_{t-1}) \quad .\end{aligned}$$

So, for any $A_t \in \mathcal{S}$, $f = 1_{S_1 \times \ldots \times S_{t-1} \times A_t}$,

$$\begin{aligned}[X_t \circ P](A_t) &= \int 1_{S_1\times\ldots\times S_{t-1}\times A_t}(x_1, \ldots, x_t)[(X_s)_{s<t} \circ P](dx_1, \ldots, dx_t) \\ &= \int_{S_{t-1}} \int_{A_t} [X_1 \circ P](dx_1 \,|\, X_1 = x_1) \cdots [X_{t-1} \circ P](dx_{t-1} \,|\, X_{t-1} = x_{t-1}) \quad .\end{aligned}$$

Thus, if we call $\lambda_t(\chi_{t-1},A_t) = \int_{A_t} [\chi_t \circ P](dx_t|[\chi_{t-1} = x_{t-1}])$, as it is easy to verify is a transition from S_{t-1} to S, and we can write:

$$[\chi_t \circ P](A_t) = (\lambda_t \circ [\chi_{t-1} \circ P])(A_t) \quad .$$

It is called the transition (or kernel) in t for the process. In the case where for every t, $\lambda_t = \lambda$, the process is said to be homogeneous and it is perfectly defined by $\chi_1 \circ P$ and λ, $\chi_t \circ P = \lambda^{t-2} \circ [\chi_1 \circ P]$.

Appendix 2

Weak Topology on a Set of Transition

All sets $A, B, C, \ldots$ are supposed metric and separable in this appendix.

If $\mathcal{T}$ is the set of transitions from A to B, we need to define a notion of convergence of a sequence which fits at the same time the notion of convergence for the sequence $T_n(\cdot, \bar{B})$ of measurable function in $[0,1]$ and the sequence $T_n(a,\cdot)$ of probability measures on B. The distance on $\mathcal{T}$ defined by $\|T\| = \sup_{a \in A} \|T(a,\cdot)\|$ is a candidate but does not give many properties to $\mathcal{T}$. In a Bayesian structure, where there is a probability measure μ on A, the general idea is to transform the problem in terms of transitions to problems in terms of measure on the set $A \times B$ (Florens [1977], Raoult [1975]). For a fixed μ, every transition T from A to B defines a unique probability on $A \times B$: $T \otimes \mu$, whose marginals are μ on A and $T \circ \mu$ on B. Reciprocally, every measure on $A \times B$ whose marginal on A is μ, defines μ-a.e. a transition π^A (π conditioned by the projection on A) such that $\pi^A \otimes \mu = T$.

If we call $\mathcal{T}'\mu$ the set of probabilities on $A \times B$ whose marginals are μ, we have defined a bijection I between $\mathcal{T}'\mu$ and the set $\mathcal{T}\mu$ of equivalence classes μ-a.e. of transitions from A to B.

<u>Definition</u>: μ-topology on a set of transition from $(A,\mathcal{A},\mu)$ to $(B,\mathcal{B})$. For every topology on $\mathcal{T}'\mu$, we shall define on $\mathcal{T}\mu$ the topology which makes I an isomorphism, which is to say:

$$T_n \xrightarrow[\text{in } \mathcal{T}\mu]{} T \qquad \mu \otimes T_n \xrightarrow[\text{in } \mathcal{T}'\mu]{} \mu \otimes T$$

and call it μ-topology.

<u>μ-weak topology on $\mathcal{T}$.</u>

We need a topology which makes $\mathcal{T}$ compact. If A and B are compact, and thus $A \times B$ is compact too, we are using on $\mathcal{T}'\mu$ the topology of the weak convergence of measures which we define by

$$\pi_n \longrightarrow \pi \quad \text{iff:}$$

$$\forall f \in \mathcal{B}_1(B), \quad \int_B f \, d\pi_n \longrightarrow \int_B f \, d\pi$$

where $\mathcal{B}_1(B)$ is the set of real valued functions on B bounded by 1. For this topology if the set $A \times B$ is compact, the set of probabilities on $A \times B$, $\mathcal{T}'$ is compact.

<u>Proposition</u>: The set $\mathcal{T}'\mu$ of probabilities on $A \times B$ compact, whose marginals on A are μ, is compact.

<u>Proof</u>: The set $\mathcal{T}'\mu$ of probabilities whose marginals are μ, is a closed set of $\mathcal{T}'$. The marginals of π is $\text{proj}_A \circ \pi$ and the main theorem of weak convergence is that if $\pi_n \longrightarrow \pi, x \circ \pi_n \longrightarrow x \circ \pi$. Here, if $\pi_n \longrightarrow \pi$, $\mu = \text{proj}_A \circ \pi_n \longrightarrow \text{proj}_A \circ \pi$ which is thus μ.

Note: Because all $\pi \in \mathcal{T}\mu$ have the same marginal μ, it is sufficient to assume that μ is tight, and B compact for $\mathcal{T}\mu$ to be compact.

We will need the following:

Proposition: For every sequence of transition T_n: $A \multimap B$, T_n converges μ-weakly toward T if and only if for every C and every transition $U: B \multimap C$, $U \circledast T_n$ converges μ-weakly to $U \circledast T$.

Proof: $U \circledast T_n \xrightarrow[\mu-w]{} U \circledast T \iff \forall\, f \in B_1(A \times B \times C)$,

$$\int_A \mu(da) \int_{B\times C} f(a,b,c) \quad U \circledast T_n(a,db,dc) \quad \text{converges.}$$

We can write the integral

$$\int_A \mu(da) \int_B \left[\int_C U(b,dc)\, f(a,b,c)\right] T_n(a,db)$$

as

$$\int_C U(b,dc)\, f(a,b,c) \in B_1(A \times B) \quad .$$

By definition of $T_n \xrightarrow[\mu-w]{} T$, the integral converges toward

$$\int_A \mu(da) \int_B \int_C U(b,dc) f(a,b,c) T(a,db)$$

which is

$$\int_A \mu(da) \int_{B\times C} f(a,b,c) \quad U \circledast T\,(a,db) \quad .$$

The converse is obvious: take $C = B$ and U = identity on B.

<u>Proposition</u>: For every sequence $U_n: B \multimap C$ and every transition $T: A \multimap B$, U_n converges $(T \circ \mu)$-weakly toward U, iff $U_n \otimes T$ converges μ-weakly toward $U \otimes T$.

<u>Proof</u>: $U_n \xrightarrow[T\circ\mu-w]{} U \Longleftrightarrow \forall\, f \in B_1(C),\ \int_B T\circ\,\mu(db) \int_C f(c)\, U_n(b,dc)$

converges $\Longleftrightarrow \int_B \int_A T(a,db)\ \mu(da) \int_C f(c)\ U_n(b,dc)$ converges. Then $U_n \otimes T \xrightarrow[\mu-w]{} U \otimes T$ iff.

$$\forall\, g \in \beta_1(B \times C),\ \int_A (da) \int_{B\times C} g(b,c) U_n(b,dc) T(a,db)$$

converges, but this is true as $\int_B g(b,c)T(a,db) \in B(C)$. The converse is evident as it is sufficient to take $g(b,c)$ constant in b to have a function of $B(C)$.

Appendix 3

Forecasts

In our model we suppose that agents know that they are in a process where the distribution on the future random events is generated by a transition $\pi^{t+1}: H^t \times A^t \longrightarrow \Omega^{t+1}$. Hence they take into account their decision when they forecast the part of Ω^{t+1} that they could know through $\sigma_i^{t+1}: H^{t+1} \longrightarrow S_i^{t+1}$. This is why we suppose the forecast function to be

$$\psi_i^{t+1}: \ S_i^t \times A_i^t \longrightarrow S_i^{t+1} \ .$$

How could this forecast function be obtained? Let us change this to $\psi_i^{t+1}: \ S_i^t \longrightarrow S_i^{t+1}$ to simplify the exposition. This anticipation function finds justification of its existence in statistical techniques of forecasting. If we suppose that uncertainty in the model can be described by a statistical structure (Ω, Θ, π) where Ω is the space of states of nature together with a σ-algebra F, Θ is the space of parameters on the probability distributions on (Ω, F). (If we suppose that Θ has a σ-algebra, we can consider π as a transition probability from Θ to Ω.) In the Bayesian case Θ, with a σ-algebra $\mathcal{E}$ has a probability measure μ. We represent:

$$\Theta \xrightarrow{\ \pi\ } \Omega \qquad\qquad \begin{tikzcd} & & S_t \\ \cdot \arrow[r, "\mu"'] & \Theta \arrow[ur, "\pi_t"] \arrow[r, "\pi"'] & \Omega \arrow[u, "q_t"'] \end{tikzcd}$$

If σ_t is the signal the agent is receiving at date t, let us call π_t the probability induced by σ_t, then $\pi_t = \sigma_t \circ \pi$ (composition must be understood in the transitional sense), that is:

$$\forall\ \theta \in \Theta,\ \forall\ \bar{S} \in S_t,\ \sigma_t \circ \pi(\theta,\bar{S}) = \int_\Omega \chi_{\sigma_i^{-1}(\bar{S})}(w)\pi(\theta,dw) = \pi(\theta,\sigma_t^{-1}(\bar{S})).$$

Now the agent is making a forecast:

(1) In a classical sense, so he has a classical non-random estimator: ε_t: $S_t \longrightarrow \Theta$ and Ψ_{t+1} can be defined by

$$\forall\ s_t \in S_t,\ \forall\ \bar{S} \in S_{t+1},\ \Psi_{t+1}(s_t,S) = \pi_t(\varepsilon_t(s_t),S) \quad .$$

Or, if ε_t is a random estimator ε_t: $S_t \multimap \Theta$, then we shall have:

$$\Psi_{t+1} = \pi_{t+1} \circ \varepsilon_t \quad ,$$

which is

$$\Psi_{t+1}(s_t,\bar{S}) = \int_\Theta \pi_{t+1}(\theta,S)\varepsilon_t(s_t,d\theta) \quad .$$

(2) In a Bayesian sense, where he deduced from his prior on Θ, a posterior distribution M_t: $S_t \multimap \Theta$, so we can define Ψ_{t+1} by:

$$\Psi_{t+1}(s_t,\bar{S}) = \int_\Theta M_t(s_t,d\theta)\pi_{t+1}(\theta,\bar{S}) = \pi_{t+1} \circ M_t(s_t,\bar{S}) \quad .$$

$$\begin{tikzcd} & S_t \arrow[dl, "\varepsilon_t"'] \\ \theta \arrow[dr, "\pi_{t+1}"'] & \\ & S_{t+1} \arrow[uu, "\Psi_{t+1}"'] \end{tikzcd} \qquad \begin{tikzcd} & & S_t \arrow[dl, "M_t"'] \\ \cdot & \theta \arrow[l, "\mu"'] \arrow[dr, "\pi_t"'] & \\ & & S_{t+1} \arrow[uu, "\Psi_{t+1}"'] \end{tikzcd}$$

A particular case is when agents have a probabilistic uncertainty: they know the probability space, but they just don't know which event is going to occur. In that case a forecast function is simply a function from S_t to S_{t+1}. It is a particular case in the sense that a measurable function is a particular case of transition. If $\Psi_{t+1}: S_t \longrightarrow S_{t+1}$ is the forecast function, the agent will maximize $U(\Psi_{t+1}(s_t,\cdot))$, which is

$$\int_{S_{t+1}} U(s) \ \Psi_{t+1}(s_t, ds)$$

as

$$\Psi_{t+1}(s_t,\cdot) = \delta_{\Psi^{-1}_{t+1}(S_t)}(\cdot) \quad (\delta \text{ is for Dirac measure.})$$

Example: It is not often that the forecast functions are explicitly given in the models reviewed, although it can be done easily, in some cases. The Cyert and DeGroot [1974] paper has a very deep statistical foundation which is appropriate. Here the signal is price. Thus price P_{t+1} in period $t + 1$, given price P_t in period t is supposed to be of the following orm: $P_{t+1} = aP_t + V_{t+1}$ where a is fixed but unknown and $(V_t)_{t\in\mathbb{N}}$ is a sequence of independent identically distributed

random errors with mean 0 and precision r. (The precision of a normal distribution is the reciprocal of its variance.) If the firm has a given posterior distribution for the value of a after it has observed the price P_t, then this distribution becomes the new prior and the firm can apply Bayes's theorem to determine the posterior distribution of a after the price P_{t+1} has been observed. In particular, it follows that if the posterior distribution of a at the end of a period t is normal with mean m_t and precision h_t, then the posterior distribution at the end of period $t + 1$ will be normal with mean m_{t+1} and precision h_{t+1}, where

$$m_{t+1} = \frac{h_t m_t + rP_t P_{t+1}}{h_t + rP_t^2}$$

and

$$h_{t+1} = h_t + rP_t^2 \quad .$$

Thus $P_{t+1} = (P_{t+1}, P_t)$, the prior $M_t = \aleph(m_t, h_t)$ and then $\Psi(P_{t+1-}) = \aleph(M_{t+1}, h_{t+1})$ as defined above.

Appendix 4

Stochastic Process of Markets

Here are two examples of stochastic processes of markets which give some insight on the Motion Law.

1. Green and Majumdar [1975] consider a market process in which agents use their private signals $s_i^t \in S_i^t$ and given prices at time t, $p^t \in \Delta^t$ to decide their demand ξ_i^t: $S_i^t \times \Delta^t \longrightarrow X_i^t$. The authors assume that an "equilibration" function h^t relates $\Delta^t \times X^t$ and Δ^{t+1}, because there is no equilibrium at date t (the total excess demand $\xi^t = \sum_{i \in I} \xi_i^t$ is not null), and h^t expresses the tendency of the system to get closer to equilibrium. Unhappily, no economic justification of this function is given and it is not clear that there is one.

The idea is to prove that the process of prices is Markovian with kernel λ and that there exists a distribution π^* on Δ such that this process is stationary $(\pi^* = \lambda \circ \pi^*)$.

Actually, λ is defined by $\lambda^t(p,\cdot) = h^t(p,\cdot) \circ \xi^t(p,\cdot) \circ \mu$ where μ is the distribution on S^t, and $\lambda^t(p,\cdot)$ is a distribution on Δ^{t+1}. The authors did not prove that if π^t is the distribution on Δ^t, $\pi^{t+1} = \lambda^t \circ \pi^t$, but, from our work, we can see this as a consequence of the model.

To prove stationarity, assumptions are to be made on ξ^t and h^t. The most important being that these two functions do not depend on time! Then λ does not depend on time either and the process is

homogeneous. Stationarity means that there exists an invariant measure π^* such that λ is neutral for the composition of transition (see Appendix 1). The authors prove the existence of a set $\Delta' \subset \Delta$ such that $\lambda(p, \Delta') = 1$ for $p \in \Delta'$.

2. Grandmont and Hildenbrand [1974] (see also Grandmont [1977], Section 5) consider a process of markets in temporary equilibria. In this process the exogenous random variable is the endowment of agents at date t which is assumed to be a homogeneous Markov process. The equilibrium state of the economy at date t depends on previous states and on the random endowments. The existence of a temporary equilibrium correspondence which relates past states and present endowment to the present state is stated. This correspondence is the link between two periods. Through a measurable selection of this correspondence, the state of the economy is proved to be a Markov process as well as the initial endowment. If the correspondence does not depend on time, this process is homogeneous and again, an invariant measure is proved to exist which makes the process stationary.

References

Allen, B. [1981], "The Existence of Expectations Equilibria in a Large Economy with Noisy Price Observation," Econometric European Meeting, Amsterdam.

Anderson, R.M. and H. Sonnenschein [1981], "On the Existence of Rational Expectations Equilibrium" Institute for Mathematical Studies in the Social Sciences Workshop Paper, Stanford University.

Anderson, R.M. and H. Sonnenschein [1981], "Rational Expectations Equilibrium with Linear Models," Princeton Research Memo No. 286, Princeton University.

Blackwell, D. [1965], "Discounted Dynamic Programming," Econometrica.

Bray, M. [1980], "Learning, Estimation, and the Stability of Rational Expectations," Stanford Graduate School of Business Res. Paper No. 550.

Cyert, R.M. and M.H. DeGroot [1974], "Rational Expectations and Bayesian Analysis," Journal of Political Economy, 82.

Florens, J.P. and M. Mouchart [1977], "Reduction of Bayesian Experiments," Université Catholique de Louvain, Core D.P. 7737.

Grandmont, J.M.[1970], "On the Temporary Equilibrium Analysis," Center for Research in Management Science, Working Paper No. 305, University of California at Berkeley.

Grandmont, J.M.[1977], "Temporary General Equilibrium Theory," Econometrica, Vol. 45, No. 3.

Grandmont, J.M. and W. Hildenbrand [1974], "Stochastic Processes of Temporary Equilibria," Journal of Mathematical Economics, 1.

Green, J. and M. Majumdar [1975], "The Nature of Stochastic Equilibria," Econometrica, 43.

Neveu, J. [1970], "Bases Mathematiques du calcul des probabilités," Masson, Paris.

Radner, R. [1974], "Equilibrium Under Uncertainty," Handbook of Mathematical Economics, edited by K. Arrow and M. Intriligator, Ch. 20, Vol. II.

Raoult, J.P. [1975], "Structures Statistiques," P.U.F., Paris.

Shefrin, H. [1980], "Games with Self-Generating Distributions," University of Santa Clara Working Paper.

REPORTS IN THIS SERIES

160. "The Structure and Stability of Competitive Dynamical Systems," by David Cass and Karl Shell.
161. "Monopolistic Competition and the Capital Market," by J. E. Stiglitz.
162. "The Corporation Tax," by J. E. Stiglitz.
163. "Measuring Returns to Scale in the Aggregate and the Scale Effect of Public Goods," by David A. Starrett.
164. "Monopoly, Quality, and Regulation," by Michael Spence.
165. "A Note on the Budget Constraint in a Model of Borrowing," by Duncan K. Foley and Martin F. Hellwig.
166. "Incentives, Risk, and Information: Notes Towards a Theory of Hierarchy," by Joseph E. Stiglitz.
167. "Asymptotic Expansions of the Distributions of Estimates in Simultaneous Equations for Alternative Parameter Sequences," by T. W. Anderson.
168. "Estimation of Linear Functional Relationships: Approximate Distributions and Connections with Simultaneous Equations in Econometrics," by T. W. Anderson.
169. "Monopoly and the Rate of Extraction of Exhaustible Resources," by Joseph E. Stiglitz.
170. "Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information," by Michael Rothschild and Joseph Stiglitz.
171. "Strong Consistency of Least Squares Estimates in Normal Linear Regression," by T. W. Anderson and John B. Taylor.
172. "Incentive Schemes under Differential Information Structures: An Application to Trade Policy," by Partha Dasgupta and Joseph Stiglitz.
173. "The Incidence and Efficiency Effects of Taxes on Income from Capital," by John B. Shoven.
174. "Distribution of a Maximum Likelihood Estimate of a Slope Coefficient: The LIML Estimate for Known Covariance Matrix," by T. W. Anderson and Takamitsu Sawa.
175. "A Comment on the Test of Overidentifying Restrictions," by Joseph B. Kadane and T. W. Anderson.
176. "An Asymptotic Expansion of the Distribution of the Maximum Likelihood Estimate of the Slope Coefficient in a Linear Functional Relationship," by T. W. Anderson.
177. "Some Experimental Results on the Statistical Properties of Least Squares Estimates in Control Problems," by T. W. Anderson and John B. Taylor.
178. "A Note on "Fulfilled Expectations" Equilibria," by David M. Kreps.
179. "Uncertainty and the Rate of Extraction under Alternative Institutional Arrangements," by Partha Dasgupta and Joseph E. Stiglitz.
180. "Budget Displacement Effects of Inflationary Finance," by Jerry Green and E. Sheshinski.
181. "Towards a Marxist Theory of Money," by Duncan K. Foley.
182. "The Existence of Futures Markets, Noisy Rational Expectations and Informational Externalities," by Sanford Grossman.
183. "On the Efficiency of Competitive Stock Markets where Traders have Diverse Information," by Sanford Grossman.
184. "A Bidding Model of Perfect Competition," by Robert Wilson.
185. "A Bayesian Approach to the Production of Information and Learning by Doing," by Sanford J. Grossman, Richard E. Kihlstrom and Leonard J. Mirman.
186. "Disequilibrium Allocations and Recontracting," by Jean-Michel Grandmont, Guy Laroque and Yves Younes.
187. "Agreeing to Disagree," by Robert J. Aumann.
188. "The Maximum Likelihood and the Nonlinear Three Stage Least Squares Estimator in the General Nonlinear Simultaneous Equation Model," by Takeshi Amemiya.
189. "The Modified Second Round Estimator in the General Qualitative Response Model," by Takeshi Amemiya.
190. "Some Theorems in the Linear Probability Model," by Takeshi Amemiya.
191. "The Bilinear Complementarity Problem and Competitive Equilibria of Linear Economic Models," by Robert Wilson.
192. "Noncooperative Equilibrium Concepts for Oligopoly Theory," by L. A. Gerard-Varet.
193. "Inflation and Costs of Price Adjustment," by Eytan Sheshinski and Yoram Weiss.
194. "Power and Taxes in a Multi-Commodity Economy," by R. J. Aumann and M. Kurz.
195. "Distortion of Preferences, Income Distribution and the Case for a Linear Income Tax," by Mordecai Kurz.
196. "Search Strategies for Nonrenewable Resource Deposits," by Richard J. Gilbert.
197. "Demand for Fixed Factors, Inflation and Adjustment Costs," by Eytan Sheshinski and Yoram Weiss.
198. "Bargains and Ripoffs: A Model of Monopolistically Competitive Price Dispersions," by Steve Salop and Joseph Stiglitz.
199. "The Design of Tax Structure: Direct Versus Indirect Taxation by A. B. Atkinson and J. E. Stiglitz.
200. "Market Allocations of Location Choice in a Model with Free Mobility," by David Starrett.
201. "Efficiency in the Optimum Supply of Public Goods," by Lawrence J. Lau, Eytan Sheshinski and Joseph E. Stiglitz.
202. "Risk Sharing, Sharecropping and Uncertain Labor Markets," by David M. G. Newberry
203. "On Non-Walrasian Equilibria," by Frank Hahn.
204. "A Note on Elasticity of Substitution Functions," by Lawrence J. Lau.
205. "Quantity Constraints as Substitutes for Inoperative Markets: The Case of the Credit Markets," by Mordecai Kurz.
206. "Incremental Consumer's Surplus and Hedonic Price Adjustment," by Robert D. Willig.
207. "Optimal Depletion of an Uncertain Stock," by Richard Gilbert.
208. "Some Minimum Chi-Square Estimators and Comparisons of Normal and LogisticaModels in Qualitative Response Analysis," by Kumio Morimune.
209. "A Characterization of the Optimality of Equilibrium in Incomplete Markets," by Sanford J. Grossman.
210. "Inflation and Taxes in a Growing Economy with Debt and Equity Finance," by M. Feldstein, J. Green and E. Sheshinski.
211. "The Specification and Estimation of a Multivariate Logit Model," by Takeshi Amemiya.
212. Prices and Queues as Screening Devices in Competitive Markets," by Joseph E. Stiglitz.
213. "Conditions for Strong Consistency of Least Squares Estimates in Linear Models," by T. W. Anderson and John B. Taylor.
214. "Utilitarianism and Horizontal Equity: The Case for Random Taxation," by Joseph E. Stiglitz.
215. "Simple Formulae for Optimal Income Taxation and the Measurement of Inequality," by Joseph E. Stiglitz.
216. "Temporal Resolution of Uncertainty and Dynamic Choice Behavior," by David M. Kreps and Evan L. Porteus.
217. "The Estimation of Nonlinear Labor Supply Functions with Taxes from a Truncated Sample," by Michael Hurd.
218. "The Welfare Implications of the Unemployment Rate," by Michael Hurd.
219. "Keynesian Economics and General Equilibrium Theory: Reflections on Some Current Debates," by Frank Hahn.
220. "The Core of an Exchange Economy with Differential Information," by Robert Wilson.
221. "A Competitive Model of Exchange," by Robert Wilson.
222. "Intermediate Preferences and the Majority Rule," by Jean-Michel Grandmont.
223. "The Fixed Price Equilibria: Some Results in Local Comparative Statics," by Guy Laroque.
224. "On Stockholder Unanimity in Making Production and Financial Decisions," by Sanford J. Grossman and Joseph E. Stiglitz.
225. "Selection of Regressors," by Takeshi Amemiya.
226. "A Note on A Random Coefficients Model," by Takeshi Amemiya.
227. "A Note on a Heteroscedastic Model," by Takeshi Amemiya.
228. "Welfare Measurement for Local Public Finance," by David Starrett.
229. "Unemployment Equilibrium with Rational Expectations," by W. P. Heller and R. M. Starr.
230. "A Theory of Competitive Equilibrium in Stock Market Economies," by Sanford J. Grossman and Oliver D. Hart.
231. "An Application of Stein's Methods to the Problem of Single Period Control of Regression Models," by Asad Zaman.
232. "Second Best Welfare Economics in the Mixed Economy," by David Starrett.
233. "The Logic of the Fix-Price Method," by Jean-Michel Grandmont.
234. "Tables of the Distribution of the Maximum Likelihood Estimate of the Slope Coefficient and Approximations," by T. W. Anderson and Takamitsu Sawa.
235. "Further Results on the Informational Efficiency of Competitive Stock Markets," by Sanford Grossman.
236. "The Estimation of a Simultaneous-Equation Tobit Model," by Takeshi Amemiya.
237. "The Estimation of a Simultaneous-Equation Generalized Probit Model," by Takeshi Amemiya.
238. "The Consistency of the Maximum Likelihood Estimator in a Disequilibrium Model," by T. Amemiya and G. Sen.
239. "Numerical Evaluation of the Exact and Approximate Distribution Functions of the Two-Stage Least Squares Estimate," by T. W. Anderson and Takamitsu Sawa.
240. "Risk Measurement of Public Projects," by Robert Wilson.
241. "On the Capitalization Hypothesis in Closed Communities," by David Starrett.
242. "A Note on the Uniqueness of the Representation of Commodity-Augmenting Technical Change," by Lawrence J. Lau.
243. "The Property Rights Doctrine and Demand Revelation under Incomplete Information," by Kenneth J. Arrow.
244. "Optimal Capital Gains Taxation Under Limited Information," by Jerry R. Green and Eytan Sheshinski.
245. "Straightforward Individual Incentive Compatibility in Large Economies," by Peter J. Hammond.
246. "On the Rate of Convergence of the Core," by Robert J. Aumann.
247. "Unsatisfactory Equilibria," by Frank Hahn.
248. "Existence Conditions for Aggregate Demand Functions: The Case of a Single Index," by Lawrence J. Lau.
249. "Existence Conditions for Aggregate Demand Functions: The Case of Multiple Indexes," by Lawrence J. Lau.
250. "A Note on Exact Index Numbers," by Lawrence J. Lau.
251. "Linear Regression Using Both Temporally Aggregated and Temporally Disaggregated Data," by Cheng Hsiao.
252. "The Existence of Economic Equilibria: Continuity and Mixed Strategies," by Partha Dasgupta and Eric Maskin.
253. "A Complete Class Theorem for the Control Problem and Further Results on Admissibility and Inadmissibility," by Asad Zaman.
254. "Measure-Based Values of Market Games," by Sergiu Hart.
255. "Altruism as an Outcome of Social Interaction," by Mordecai Kurz.
256. "A Representation Theorem for 'Preference for Flexibility'," by David M. Kreps.
257. "The Existence of Efficient and Incentive Compatible Equilibria with Public Goods," by Theodore Groves and John O. Ledyard.
258. "Efficient Collective Choice with Compensation," by Theodore Groves.
259. "On the Impossibility of Informationally Efficient Markets," by Sanford J. Grossman and Joseph E. Stiglitz.

REPORTS IN THIS SERIES

260. "Values for Games Without Sidepayments: Some Difficulties With Current Concepts," by Alvin E. Roth.
261. "Martingales and the Valuation of Redundant Assets," by J. Michael Harrison and David M. Kreps.
262. "Autoregressive Modelling and Money — Income Causality Detection," by Cheng Hsiao.
263. "Measurement Error in a Dynamic Simultaneous Equations Model without Stationary Disturbances," by Cheng Hsiao.
264. "The Measurement of Deadweight Loss Revisited," by W. E. Diewert.
265. "The Elasticity of Derived Net Supply and a Generalized Le Chatelier Principle," by W. E. Diewert.
266. "Income Distribution and Distortion of Preferences: the ℓ Commodity Case," by Mordecai Kurz.
267. "The n^{-2} Order Mean Squared Errors of the Maximum Likelihood and the Minimum Logit Chi-Square Estimator," by Takeshi Amemiya.
268. "Temporal Von Neumann-Morgenstern and Induced Preferences," by David M. Kreps and Evan L. Porteus.
269. "Take-Over Bids and the Theory of the Corporation," by Stanford Grossman and Oliver D. Hart.
270. "The Numerical Values of Some Key Parameters in Econometric Models," by T. W. Anderson, Kimio Morimune and Takamitsu Sawa.
271. "Two Representations of Information Structures and their Comparisons," by Jerry Green and Nancy Stokey.
272. "Asymptotic Expansions of the Distributions of Estimators in a Linear Functional Relationship when the Sample Size is Large," by Naoto Kunitomo.
273. "Public Goods and Power," by R. J. Aumann, M. Kurz and A. Neyman.
274. "An Axiomatic Approach to the Efficiency of Non-Cooperative Equilibrium in Economies with a Continuum of Traders," by A. Mas-Colell.
275. "Tables of the Exact Distribution Function of the Limited Information Maximum Likelihood Estimator when the Covariance Matrix in Known," by T. W. Anderson and Takamitsu Sawa.
276. "Autoregressive Modeling of Canadian Money and Income Data," by Cheng Hsiao.
277. "We Can't Disagree Forever," by John D. Geanakoplos and Heraklis Polemarchakis.
278. "Constrained Excess Demand Functions," by Herklis M. Polemarchakis.
279. "On the Bayesian Selection of Nash Equilibrium," by Akira Tomioka.
280. "Disequilibrium Econometrics in Simultaneous Equations Systems," by C. Gourieroux, J. J. Laffont and A. Monfort.
281. "Duality Approaches to Microeconomic Theory," by W. E. Diewert.
282. "A Time Series Analysis of the Impact of Canadian Wage and Price Controls," by Cheng Hsiao and Oluwatayo Fakiyesi.
283. "A Strategic Theory of Inflation," by Mordecai Kurz.
284. "A Characterization of Vector Measure Games in pNA," by Yair Tauman.
285. "On the Method of Taxation and the Provision of Local Public Goods," by David A. Starrett.
286. "An Optimization Problem Arising in Economics: Approximate Solutions, Linearity, and a Law of Large Numbers," by Sergiu Hart.
287. "Asymptotic Expansions of the Distributions of the Estimates of Coefficients in a Simultaneous Equation System," by Yasunori Fujikoshi, Kimio Morimune, Naoto Kunitomo and Masanobu Taniguchi.
288. "Optimal & Voluntary Income Distribution," by K. J. Arrow.
289. "Asymptotic Values of Mixed Games," by Abraham Neyman.
290. "Time Series Modelling and Causal Ordering of Canadian Money, Income and Interest Rate," by Cheng Hsiao.
291. "An Analysis of *Power in Exchange Economies*," by Martin J. Osborne.
292. "Estimation of the Reciprocal of a Normal Mean," by Asad Zaman.
293. "Improving the Maximum Likelihood Estimate in Linear Functional Relationships for Alternative Parameter Sequences," by Kimio Morimune and Naoto Kunitomo.
294. "Calculation of Bivariate Normal Integrals by the Use of Incomplete Negative-Order Moments," by Kei Takeuchi and Akimichi Takemura.
295. "On Partitioning of a Sample with Binary-Type Questions in Lieu of Collecting Observations," by Kenneth J. Arrow, Leon Pesotchinsky and Milton Sobel.
296.
297. "The Two Stage Least Absolute Deviations Estimators," by Takeshi Amemiya.
298. "Three Essays on Capital Markets," by David M. Kreps.
299. "Infinite Horizon Programs," by Michael J. P. Magill.
300. "Electoral Outcomes and Social Log-Likelihood Maxima," by Peter Coughlin and Shmuel Nitzan.
301. "Notes on Social Choice and Voting," by Peter Coughlin.
302. "Overlapping Expectations and Hart's Conditions for Equilibrium in a Securities Model," by Peter J. Hammond.
303. "Directional and Local Electorate Competitions with Probabilistic Voting," by Peter Coughlin and Shmuel Nitzan.
304. "Asymptotic Expansions of the Distributions of the Test Statistics for Overidentifying Restrictions in a System of Simultaneous Equations," by Kunitomo, Morimune, and Tsukuda.
305. "Incomplete Markets and the Observability of Risk Preference Properties," by H. H. Polemarchakis and L. Selden.
306. "Multi-Period Securities and the Efficient Allocation of Risk: A Comment on the Black-Scholes Option Pricing Model," by David M. Kreps.
307. "Asymptotic Expansions of the Distributions of k-Class Estimators when the Disturbances are Small," by Naoto Kunitomo, Kimio Morimune, and Yoshihiko Tsukuda.
308. "Arbitrage and Equilibrium in Economies with Infinitely Many Commodities," by David M. Kreps.
309. "Unemployment Equilibrium in an Economy with Linked Prices," by Mordecai Kurz.
310. "Pareto Optimal Nash Equilibria are Competitive in a Repeated Economy," by Mordecai Kurz and Sergiu Hart.
311. "Identification," by Cheng Hsiao.
312. "An Introduction to Two-Person Zero Sum Repeated Games with Incomplete Information," by Sylvain Sorin.
313. "Estimation of Dynamic Models with Error Components," by T. W. Anderson and Cheng Hsiao.
314. "On Robust Estimation in Certainty Equivalence Control," by Anders H. Westlund and Hans Stenlund.
315. "On Industry Equilibrium Under Uncertainty," by J. Dreze and E. Sheshinski.
316. "Cost Benefit Analysis and Project Evaluation from the Viewpoint of Productive Efficiency," by W. E. Diewert.
317. "On the Chain-Store Paradox and Predation: Reputation for Toughness," by D. M. Kreps and Robert Wilson.
318. "On the Number of Commodities Required to Represent a Market Games," Sergiu Hart.
319. "Evaluation of the Distribution Function of the Limited Information Maximum Likelihood Estimator," by T. W. Anderson, Naoto Kunitomo, and Takamitsu Sawa.
320. "A Comparison of the Logit Model and Normal Discriminant Analysis when the Independent Variables are Binary," by Takeshi Amemiya and James L. Powell.
321. "Efficiency of Resource Allocation by Uninformed Demand," by Theodore Groves and Sergiu Hart.
322. "A Comparison of the Box-Cox Maximum Likelihood Estimator and the Nonlinear Two Stage Least Squares Estimator," by Takeshi Amemiya and James L. Powell.
323. "Comparison of the Densities of the TSLS and LIMLK Estimators for Simultaneous Equations," by T. W. Anderson, Naoto Kunitomo, and Takamitsu Sawa.
324. "Admissibility of the Bayes Procedure Corresponding to the Uniform Prior Distribution for the Control Problem in Four Dimensions but Not in Five," by Charles Stein and Asad Zaman.
325. "Some Recent Developments on the Distributions of Single-Equation Estimators," by T. W. Anderson.
326. On Inflation," by Frank Hahn.
327. Two Papers on Majority Rule: "Continuity Properties of Majority Rule with Intermediate Preferences," by Peter Coughlin and Kuan-Pin Lin; and, "Electoral Outcomes with Probabilistic Voting and Nash Social Welfare Maxima," by Peter Coughlin and Shmuel Nitzan.
328. "On the Endogenous Formation of Coalitions," by Sergiu Hart and Mordecai Kurz.
329. "Controllability, Pecuniary Externalities and Optimal Taxation," by David Starrett.
330. "Nonlinear Regression Models," by Takeshi Amemiya.
331. "Paradoxical Results from Inada's Conditions for Majority Rule," by Herve Raynaud.
332. "On Welfare Economics with Incomplete Information and the Social Value of Public Information," by Peter J. Hammond.
333. "Equilibrium Policy Proposals with Abstentions," by Peter J. Coughlin.
334. "Infinite Excessive and Invariant Measures," by Michael I. Taksar.
335. "The Life-Cycle Hypothesis and the Effects of Social Security and Private Pensions on Family Savings," by Mordecai Kurz.
336. "Optimal Retirement Age," by Mordecai Kurz.
337. "Bayesian Incentive Compatible Beliefs," by Claude d'Aspremont and Louis-Andre Gerard-Varet.
338. "Qualitative Response Models: A Survey," by Takeshi Amemiya.
339. "The Social Costs of Monopoly and Regulation: A Game Theoretic Analysis," by William P. Rogerson.
340. "Sequential Equilibria," by David M. Kreps and Robert Wilson.
341. "Enhancing of Semigroups," by Michael I. Taksar.
342. "Formulation and Estimation of Dynamic Models Using Panel Data," by T. W. Anderson and Cheng Hsiao.
343. "Ex-Post Optimality as a Dynamically Consistent Objective for Collective Choice Under Uncertainty," by Peter Hammond.
344. "Three Lectures in Monetary Theory," by Frank H. Hahn.
345. "Socially Optimal Investment Rules in the Presence of Incomplete Markets and Other Second Best Distortions," by Frank Milne and David A. Starrett.
346. "Approximate Purification of Mixed Strategies," by Robert Aumann, Yitzhak Katznelson, Roy Radner, Robert W. Rosenthal, and Benjamin Weiss.
347. "Conditions for Transitivity of Majority Rule with Algorithmic Interpretations," by Herve Raynaud.
348. "How Restrictive Actually are the Value Restriction Conditions," by Herve Raynaud.
349. "Cournot Duopoly in the Style of Fulfilled Expectations Equilibrium," by William Novshek and Hugh Sonnenschein.
350. "Law of Large Numbers for Random Sets and Allocation Processes," by Zvi Artstein and Sergiu Hart.
351. "Risk Perception in Psychology and Economics," by Kenneth J. Arrow.
352. "Shrunken Predictors for Autoregressive Models," by Taku Yamamoto.
353. "Predation, Reputation, and Entry Deterrence," by Paul Milgrom and John Roberts.
354. "Social and Private Production Objectives in the Sequence Economy," by David Starrett.
355. "Recursive Rational Choice," by Alain Lewis.
356. "Least Absolute Deviations Estimation for Censored and Truncated Regression Models," by James Powell.
357. "Relatively Recursive Rational Choice," by Alain Lewis.
358. "A Theory of Auctions and Competitive Bidding," by Paul Milgrom and Robert Weber.
359. "Facing an Uncertain Future," by W. M. Gorman.
360. "The Individual Freedom Allowed by the Value Restriction Condition," by Herve Raynaud.
361. "Incomplete Resource Allocation Mechanisms," by P. J. Hammond.
362. "The Comparative Dynamics of Efficient Programs of Capital Accumulation and Resource Depletion," by W. E. Diewert and T. R. Lewis.
363. "Optimum Pricing Policy Under Stochastic Inflation," by E. Sheshinski and Y. Weiss.
364. "Capital Accumulation and the Characteristics of Private Intergenerational Transfers," by M. Kurz.
365. "Asymptotic Efficiency and Higher Order Efficiency of the Limited Information Maximum Likelihood Estimator in Large Econometric Models," by N. Kunitomo.
366. "Semi-Values of Political Economic Games," by A. Neyman.
367. "Non-Zero-Sum Two-Person Repeated Games with Incomplete Information," by S. Hart.
368. "Partially Generalized Least Squares and Two-Stage Least Squares and Two-Stage Least Squares Estimators," by T. Amemiya.
369. "Lipschitz Properties of Solution in Mathematical Programming," by B. Cornet and G. Laroque.
370. "Multiple Hypothesis Testing," by N. E. Savin.
371. "Team Theory and Decentralized Resource Allocation: An Example," by K. J. Arrow.
372. "Testing for Unit Roots: 2," by G. B. A. and Evans and N. E. Savin.
373. "Asymptotic Bias of the Least Squares Estimator for Multivariate Autoregressive Models," by T. Yamamoto and N. Kunitomo.
374. "The Asymptotic Normality of Two-Stage Least Absolute Deviations Estimators," by J. Powell.
375. "Rational Cooperation in the Finitely-Repeated Prisoner's Dilemma," by D. Kreps, P. Milgrom, J. Roberts, and R. Wilson.

Reports in this Series

376. "Necessary and Sufficient Conditions for Single-Peakedness Along a Linearly Orderd Set of Policy Alternatives" by P.J. Coughlin and M.J. Hinich

377. "The Role of Reputation in a Repeated Agency Problem Involving Information Transmission" by W. P. Rogerson

378. "Unemployment Equilibrium with Stochastic Rationing of Supplies" by Ho-mou Wu

379. "Optimal Price and Income Regulation Under Uncertainty in the Model with One Producer" by M. I. Taksar

380. "On the NTU Value" by R.J. Aumann

381. "Best Invariant Estimation of a Direction Parameter with Application to Linear Functional Relationships and Factor Analysis" by T.W. Anderson, C. Stein, and A. Zaman

382. "Informational Equilibrium" by R. Kast